Topology SoSe2022

ProphetX

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Abstract

The purpose of this writing is to fill in detailed proofs for the remarks done in the Topology lecture taught by Professor Ulrich Bauer, Nico Stucki and Sebastian Spindler in SoSe2022. Therefore, the style of writing assumes the reader has read the lecture notes of the corresponding week.

Contents

1 Handout1

In the lecture notes of this week, we discussed metric spaces, topological spaces, how we specify a topology, how is a continuous function between two topological spaces defined, how we axiomatize topological spaces via neighbourhoods, open and closed maps, interior boundary, closure and countability and separability of a topological spaces. We will fill out proofs for the remarks done in the lecture.
Beware: The style of writing is rather detailed and sometimes maybe annoying for a pure mathematician. This is due to the reason that I am a physicist, who has to look up several things, while doing the proofs in a precise manner, I sometimes lack prerequisites.
If you find any typos, please let me know at lehel@csillag.ro.

I am open to any feedback regarding the writing/formatting style: (should I use more symbols, less text? should I use less symbols, more text? should I try to arrange things differently on the page?)

1.1 Continuity

Firstly, we said that topological spaces form a category, which requires that the composition of continuous maps is continuous. The continuous maps in the category Top are the morphisms.

Proposition 1.1. *Let $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y),(Z,\mathcal{O}_Z)$ be topological spaces and $f:X\to Y, g:Y\to Z$ be continuous maps with respect to $\mathcal{O}_X,\mathcal{O}_Y,\mathcal{O}_z$. Then, $g\circ f$ is continuous.*

Proof. $g\circ f$ is continuous $\iff$ $(g\circ f)^{-1}(V)\in\mathcal{O}_X\ \forall\ V\in\mathcal{O}_Z$. To show this, let $V\in\mathcal{O}_Z$ arbitrary. Then:

$$(g\circ f)^{-1}(V)=\{x\in X|(g\circ f)(x)\in V\}=\{x\in X|f(x)\in g^{-1}(V)\}=f^{-1}\left(g^{-1}(V)\right).$$

But since $V\in\mathcal{O}_Z$ and g is continuous we have that $g^{-1}(V)\in\mathcal{O}_Y$. Moreover, using that f is continuous, we have that $f^{-1}(g^{-1}(V))\in\mathcal{O}_X$, since $g^{-1}(V)\in\mathcal{O}_Y$. Thus, preimage of every open set in Z is open in X, as desired. This finishes the proof. □

1.2 Basis, neighbourhood basis

Proposition 1.2. *Let $(X,\mathcal{O}_X)$ be a topological space and B a basis of $(X,\mathcal{O}_X)$. Then $B_x:=\{A\in B|x\in A\}$ is a neighbourhood basis of x.*

Proof. Let B be a basis for $\mathcal{O}_X$. Then, every open set U in $(X,\mathcal{O}_X)$ is a union of elements of B. Let $x\in X$. Let V be a neighbourhood of x. By definition of neighbourhood, we have that there exists an open set $O\in\mathcal{O}_X:O\subseteq V, x\in O$. Since B is a basis and B_x contains those basis elements, which contain x, we have that O must be union of some elements of B_x. Therefore, there exists $A\in B_x:A\subseteq O\subseteq V$, as desired. □

Proposition 1.3. *Let $(X,\mathcal{O}_X)$ be a topological space and let B_x be a neighbourhood basis for each $x\in X$. Then $\bigcup\limits_{x\in X}B_x$ is a basis of $(X,\mathcal{O}_X)$.*

Proof. Let $B=\bigcup\limits_{x\in X}B_x$ and $U\in\mathcal{O}_X$ be an open set. We want to show that U is a union of elements of B. To this end, let $x\in U$. Then, U is a neighbourhood of x. Since B_x is a neighbourhood basis of x, we have that:

$$\exists U_x\in B_x:U_x\subseteq U\forall x\in U.$$

In particular, $U_x\in B_x$. We now claim that $U=\bigcup\limits_{x\in U}U_x$. This is true, since: $x\in U_x\implies U\subseteq\bigcup\limits_{x\in U}U_x$. Moreover, we have that $\bigcup\limits_{x\in U}U_x\subseteq U$: Let $a\in\bigcup\limits_{x\in U}\implies a\in U_X$ for some $x\in U$, but $U_x\subseteq U$, thus $a\in U$. So, we have that $U\subseteq\bigcup\limits_{x\in U}U_x\wedge\bigcup\limits_{x\in U}U_x\subseteq U\iff U=\bigcup\limits_{x\in U}U_x$. Since $U_x\in B_x$, we have that every open set $U\in\mathcal{O}_X$ can be obtained by union of elements of B, precisely the elements U_x, as desired. □

1.3 Interior, boundary, closure

Proposition 1.4. *Let $(X, \mathcal{O}_X)$ be a topological space, $A \subseteq X$ and $x \in A$. Then x is an isolated point of A iff it is not a limit point of A.*

Proof. "$\Rightarrow$ Suppose $x \in X$ is an isolated point of A. Then, we have that $\exists V \in N_x : V \cap A = \{x\}$. This implies: $\exists V \in N_x : (V \cap A) \setminus \{x\} = \varnothing$. Thus, x is not a limit point of A.
"$\Leftarrow$" Suppose $x \in X$ is not a limit point of A. Then, we have that $\exists V \in N_x : (V \cap A) \setminus \{x\} = \varnothing$. Now, we take union with x, we have:

$$\exists V \in N_x : ((V \cap A) \setminus \{x\}) \cup \{x\} = \{x\}.$$

The LHS can be equivalently rewritten as:

$$\begin{aligned} ((V \cap A) \setminus \{x\}) \cup \{x\} &= ((V \cap A) \cup \{x\}) \setminus (\{x\} \setminus \{x\}) \\ &= ((V \cap A) \cup \{x\}) \setminus \varnothing \\ &= (V \cap A) \cup \{x\} \neq V \cap A \text{ in general.} \end{aligned}$$

Therefore:

$$\exists V \in N_x : ((V \cap A) \setminus \{x\}) \cup \{x\} = \{x\} \iff \exists V \in N_x : (V \cap A) \cup \{x\} = \{x\} \iff \exists V \in N_x : V \cap A \subseteq \{x\}.$$

So far, we have $V \cap A \subseteq \{x\}$. Now, let $x \in \{x\}$. Since $V \in N_x$, we have that$x \in V$. By assumption $x \in A$. Thus $x \in V \cap A$. This implies that $\{x\} \subseteq V \cap A$. Therefore:

1. $V \cap A \subseteq \{x\}$;
2. $\{x\} \subseteq V \cap A$.

Which finally says that $V \cap A = \{x\}$. Namely, we found $V \in N_x : V \cap A = \{x\}$, that is, x is an isolated point of A. □

Remark 1.5. *We have to be careful. In general $(A \setminus B) \cup B \neq A$. Counterexample: let A, B be disjoint, B non-empty. Then $A \setminus B = A$. Thus, $(A \setminus B) \cup B = A \cup B \neq A$, since B is non-empty by assumption.*

Proposition 1.6. *Let $(X, \mathcal{O}_X)$ be a topological space and $A \subseteq X$. A point $x \in A$ is an interior point of A iff it is not a boundary point of A.*

Proof. "$\Rightarrow$" Suppose $x \in A$ is an interior point of $A \implies A \in N_x$. Then, trivially, it is not a boundary point.
"$\Leftarrow$" Suppose $x \in A$ is not a boundary point of A. Then $A \in N_x$ or $X \setminus A \in N_x$. But, by assumption, we have that $x \in A \implies A \in N_x \implies x$ is interior point of A. □

Proposition 1.7. *Let $(X, \mathcal{O}_X)$ be a topological space and $A \subseteq X$. Then $x \in X$ is a closure point of A iff it is an interior point of A or it is a boundary point of A.*

Proof. "$\Rightarrow$" Let $x \in X$ be a closure point. Then, $\forall V \in N_X : V \cap A \neq \varnothing$. Suppose $x \in X$ is not an interior point of $A \implies A \notin N_x$. But if $A \notin N_x \implies X \setminus A \notin N_x$, since all neighbourhoods $V \in N_x$ intersect A. Hence, we showed: if $x \in X$ is not an interior point of A, then $A \notin N_x$ and $X \setminus A \notin N_x$. This implies $x \in X$ is a boundary point of A.
"$\Leftarrow$ Conversely, let $x \in X$ be an interior point $\implies A \in N_x$. Thus, $x \in A$. Then, $\forall V \in n_x : V \cap A \neq \varnothing$, since $x \in V$ and $x \in A$. Thus, x is a closure point of A.
Let $x \in X$ be a boundary point $\implies X \setminus A \notin N_x \implies \forall V \in N_x : V$ can't be subset of $X \setminus A \implies V \cap A \neq \varnothing \implies x$ is a closure point of A. $\square$

Remark 1.8. *In the above proof, we used: $if B \implies C \wedge D$, then $B \implies D$ in particular. Namely, we did not use that boundary $\implies A \notin N_x$.*

Remark 1.9. *If $A \notin N_x$, it does not necessarily imply that $A \cap V = \varnothing$, where $V \in N_x$. It simply means that A doesn't contain an open set containing x, but it can still contain x.*

Proposition 1.10. *Let $(X, \mathcal{O}_X)$ be a topological space and $A \subseteq X$. Then $x \in X$ is a closure point of A iff it is either a limit point of A or an isolated point of A.*

Proof. "$\Rightarrow$ Let $x \in X$ be a closure point of A. Then, $\forall V \in N_x : V \cap A \neq \varnothing$. Suppose $x \in X$ is not an isolated point of A. This implies $\forall V \in N_x : V \cap A \neq \{x\}$. So, we have two conditions:

1. $\forall V \in N_x : V \cap A \neq \varnothing$;

2. $V \cap A \neq \{x\}$.

$V \cap A \neq \varnothing \implies \exists a \in V \cap A$. Moreover, $V \cap A \neq \{x\} \implies \exists a \in V \cap A : a \neq \{x\} \implies \exists a \in (V \cap A) \setminus \{x\} \neq \varnothing$. Finally, this shows that $(V \cap A) \setminus \{x\} \neq \varnothing, \forall V \in N_x$. Thus, x is a limit point of A.
"$\Leftarrow$" Let $x \in X$ be a limit point of A. This implies that $\forall V \in N_x : (V \cap A) \setminus \{x\} \neq \varnothing$. Moreover:

$$\forall V \in N_x : (V \cap A) \setminus \{x\} \neq \varnothing \implies \forall V \in N_x : \exists a \in V \cap A : a \neq \{x\} \implies \forall V \in N_x : \exists a \in V \cap A \implies \forall V \in N_x : V \cap A \neq \varnothing.$$

Since $\forall V \in N_x : V \cap A \neq \varnothing$, we have that x is a closure point of A.
Let $x \in X$ be an isolated point of A. This implies that $\exists U \in N_x : (U \cap A) = \{x\} \implies x \in A$. But since $x \in A$, we have that $\forall V \in N_x : V \cap A \neq \varnothing$, because $x \in V$ and $x \in A$. Thus, x is a closure point of A. $\square$

1.4 Countability

Proposition 1.11. *Let $(X, \mathcal{O}_X)$ be a second countable topological space. Then it is also first countable.*

Proof. Assume $(X, \mathcal{O}_X)$ is second countable. This implies that it has a countable basis B. By proposition 1.2, we have for each $x \in X$ a neighbourhood basis given by:

$$B_x := \{A \in B | x \in A\}.$$

We know that B is countable, because $(X, \mathcal{O}_X)$ is second countable and that $B_x \subseteq B$. Using that the subset of a countable set is countable, we have obtained a countable neighbourhood basis for each $x \in X$. Therefore, $(X, \mathcal{O}_X)$ is second countable. □

Proposition 1.12. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is first countable, then so is $(Y, \mathcal{O}_Y)$.*

Proof. As $(X, \mathcal{O}_X)$ is first countable, every point $x \in X$ has a countable neighbourhood basis $\mathcal{B}_x$. Moreover, by the definition of neighbourhood basis:

$$\forall U \in \mathcal{N}_x : \exists B \in \mathcal{B}_x : B \subseteq U.$$

By the definition of neighbourhood, we can also conclude:

$$\forall U' \in \mathcal{O}_X : x \in U' : \exists B' \in \mathcal{B}_x \cap \mathcal{O}_X : B' \subseteq U'$$

Since each $B' \in \mathcal{B}_x \cap \mathcal{O}_X$ is open in $(X, \mathcal{O}_X)$, such that $x \in X$ and f is a homeomorphism, we have that $f(B')$ is open in $(Y, \mathcal{O}_Y)$ and contains $f(x)$. Moreover, as f is bijective, we have that $\forall y \in Y : \exists x \in X : f(x) = y$. Hence, consider the following collection of open sets:

$$\mathcal{B}_y := \{f(B') | B' \in \mathcal{B}_x \cap \mathcal{O}_X\}.$$

We claim that the collection $\mathcal{B}_y$ is a countable neighbourhood basis for each $y \in Y$. Clearly, $\mathcal{B}_y$ is countable, as $\mathcal{B}_x$ was countable and f is a bijection. We now want to show that it is a neighbourhood basis. To this end, we proceed by contrapositive. Suppose $\mathcal{B}_y$ is not a neighbourhood basis of $y \in Y$. This implies:

$$\exists V \in \mathcal{N}_y : \forall M \in \mathcal{B}_y : B \not\subseteq V.$$

Writing out concretely who the elements of $\mathcal{B}_y$ are:

$$\exists V \in \mathcal{N}_y : \forall f(B') : B' \in \mathcal{B}_x \cap \mathcal{O}_X : f(B') \not\subseteq V.$$

As f is bijective:

$$\exists V \in \mathcal{N}_y : \forall B' \in \mathcal{B}_x \cap \mathcal{O}_X : B' \not\subseteq f^{-1}(V).$$

However, as $V \in \mathcal{N}_y$, there exists $V' \in \mathcal{O}_Y \cap \mathcal{N}_y : y \in V'$. Moreover, $y = f(x)$ by surjectivity, thus $f^{-1}(y) = x$. But as $y \in V'$, we have $f^{-1}(y) \in f^{-1}(V')$, and as f is a homeomorphism, $f^{-1}(V')$ is open in $(X, \mathcal{O}_X)$ and contains $f^{-1}(y) = f^{-1}(f(x)) = x$, such that there does not exist an element A in $\mathcal{B}_x \cap \mathcal{O}_X : A \subseteq f^{-1}(V')$. However, this contradicts the fact that $\mathcal{B}_x$ is a neighbourhood basis of x.

Hence, we conclude that $\mathcal{B}_y$ is a countable neighbourhood basis of $(Y, \mathcal{O}_Y)$. $\square$

Corollary 1.13. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : X \to Y$ be a homeomorphism. X is first countable iff Y is.*

Corollary 1.14. *First countability is a topological property.*

Proposition 1.15. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is second countable, then so is $(Y, \mathcal{O}_Y)$.*

Proof. Let $(X, \mathcal{O}_X)$ be second countable. Then, there exists a countable basis $\mathcal{B}$ of $(X, \mathcal{O}_X)$. Hence, every open set $U \in \mathcal{O}_X$ can be written as union of basis elements:

$$\forall U \in \mathcal{O}_X : U = \bigcup_{B \in \mathcal{B}} B.$$

This implies:

$$f(U) = f\left(\bigcup_{B \in \mathcal{B}} B\right) = \bigcup_{B \in \mathcal{B}} f(B).$$

As f is a homeomorphism, $f(U) \in \mathcal{O}_Y$. We now claim that the collection

$$\widetilde{\mathcal{B}} := \{f(B) | B \in \mathcal{B}\}$$

is a countable basis for $(Y, \mathcal{O}_Y)$. Clearly $\widetilde{\mathcal{B}}$ is countable, as $\mathcal{B}$ is countable and f is bijective. We now show it is a basis for the topology $\mathcal{O}_Y$. We proceed by contrapositive. Suppose it is not a a basis for the topology $\mathcal{O}_Y$. This implies:

$$\exists V \in \mathcal{O}_Y : V \neq \bigcup_{A \in \widetilde{\mathcal{B}}} A \iff \exists V \in \mathcal{O}_Y : V \neq \bigcup_{M \in \mathcal{B}} f(M).$$

Using that f is a homeomorphism, i.e. bijective and in both directions continuous:

$$\exists f^{-1}(V) \in \mathcal{O}_X : f^{-1}(V) \neq \bigcup_{A \in \widetilde{\mathcal{B}}} f^{-1}(A) \iff \exists f^{-1}(V) \in \mathcal{O}_X : V \neq \bigcup_{M \in \mathcal{B}} M.$$

However, this contradicts that $\mathcal{B}$ is a basis for $(X, \mathcal{O}_X)$. Hence, $\widetilde{\mathcal{B}}$ is a basis for $\mathcal{O}_Y$. $\square$

Corollary 1.16. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : X \to Y$ be a homeomorphism. X is second countable iff Y is.*

Corollary 1.17. *Second countability is a topological property.*

Proposition 1.18. *Let $(X, \mathcal{O}_X)$ be a second countable topological space. Then any subset $A \subseteq X$*

equipped with the subspace topology

$$\mathcal{O}_A := \{A \cap U | U \in \mathcal{O}_X\}$$

is a secound countable topological space.

Proof. Since $(X, \mathcal{O}_X)$ is second countable, it has a countable basis $\mathcal{B}$. We claim that

$$\mathcal{B}_A := \{A \cap S | S \in \mathcal{B}\}$$

is a countable basis of $(A, \mathcal{O}_A)$. Clearly, as $\mathcal{B}$ is countable and $\mathcal{B}_A \subseteq \mathcal{B}$, we have that $\mathcal{B}_A$ is countable aswell. We now want to show that $\mathcal{B}_A$ is a basis of $(A, \mathcal{O}_A)$. To this end, we have to show that every open set in $(A, \mathcal{O}_A)$ can be written as a union of elements of $\mathcal{B}_A$. First, note that all elements of $\mathcal{B}_A$ are open in $(A, \mathcal{O}_A)$ by the definition of subspace topology, as $\mathcal{B}$ is a basis of $(X, \mathcal{O}_X)$, so its elements are open sets. Now, let $U \subseteq A$ be open in $(A, \mathcal{O}_A)$. By the definition of subspace topology, we have:

$$U = A \cap V \quad for\, some \;\; V \subseteq X, V \in \mathcal{O}_X.$$

Since $V \in \mathcal{O}_X$ and $\mathcal{B}$ is a basis, we have that there exists a subcollection $\mathcal{B}' \subseteq \mathcal{B}$, such that:

$$V = \bigcup_{S \in \mathcal{B}'} S.$$

Hence U can be written as:

$$U = A \cap \left(\bigcup_{S \in \mathcal{B}'} S \right) = \bigcup_{S \in \mathcal{B}'} (A \cap S).$$

But clearly we have that $A \cap S \in \mathcal{O}_A$. Moreover, union of open sets is open by the definition of topology. Hence, every open set $U \in \mathcal{O}_A$ can be expressed as union of elements in $\mathcal{B}_A$, which shows that it is a basis for the topology $\mathcal{O}_A$. □

Corollary 1.19. *Second countability is hereditary.*

1.5 Separability as a topological property

Proposition 1.20. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is separable, then so is $(Y, \mathcal{O}_Y)$.*

Proof. As $(X, \mathcal{O}_X)$ is separable, there exists $A \subseteq X$, which is both countable and dense, that is:

$$\forall U \in \mathcal{O}_X : A \cap U \neq \varnothing.$$

We now claim that $f(A)$ is countable and dense in $(Y, \mathcal{O}_Y)$. Clearly, it is separable as A is countable and f is bijective. Now, let $V \in \mathcal{O}_Y$. Since f is continuous, we have that $f^{-1}(V) \in \mathcal{O}_X$. Since A is

dense, we have:

$$f^{-1}(V) \cap A \neq \varnothing.$$

Therefore:

$$f\left(f^{-1}(V) \cap A\right) \neq \varnothing \iff f(f^{-1}(V) \cap f(A) \neq \varnothing \iff V \cap f(A) \neq \varnothing.$$

As this holds for all open sets $V \in \mathcal{O}_Y$, we have that $f(A)$ is dense in $(Y, \mathcal{O}_Y)$, which finishes the proof. □

Corollary 1.21. *Separability is a topological property.*

To do: Prove separability is not hereditary.

To do: Prove separability is equivalent to second countability in metric spaces. This is in particular important for Hilbert spaces!

2 Handout2

2.1 Subspaces and induced topology

Proposition 2.1. *Let $(X, \mathcal{O}_X)$ be a topological space and let $A \subseteq X$. The subspace topology on A defined by:*

$$\mathcal{O}_A := \{A \cap U | U \in \mathcal{O}_X\}$$

is a topology on A.

Proof. We have to verify all the three axioms of a topology:

1. $\varnothing \in \mathcal{O}_A$, since $\varnothing \in \mathcal{O}_X$ and $A \cap \varnothing = \varnothing$. Similarly, we have that $A \in \mathcal{O}_A$, since $X \in \mathcal{O}_X$ and $A \cap X = A$;

2. Let $S, T \in \mathcal{O}_A$. Then, we have:

$$S \in \mathcal{O}_A \implies \exists U \in \mathcal{O}_X : S = U \cap A;$$

$$T \in \mathcal{O}_A \implies \exists V \in \mathcal{O}_X : T = V \cap A.$$

   Therefore:

$$S \cap T = (U \cap A) \cap (V \cap A) = (U \cap V) \cap A.$$

   But since $U, V \in \mathcal{O}_X$, we have that $U \cap V \in \mathcal{O}_X$, because $\mathcal{O}_X$ is a topology. Hence, by denoting $M := U \cap V \in \mathcal{O}_X$, we can rewrite the statement as:

$$S \cap T = M \cap A, \; M \in \mathcal{O}_X,$$

   which tells us precisely that $S \cap T \in \mathcal{O}_A$, by the definition of the subspace topology.

3. Let $(S_i)_{i\in I} \in \mathcal{O}_A$ be an arbtirary collection of open sets. Then, we have:

$$\forall i \in I : \exists V_i \in \mathcal{O}_X : S_i = V_i \cap A.$$

By using set theoretic arguments:

$$\bigcup_{i\in I} S_i = \bigcup_{i\in I} (V_i \cap A) = \left(\bigcup_{i\in I} V_i\right) \cap A.$$

But since each $(V_i)_{i\in I} \in \mathcal{O}_X$ and $\mathcal{O}_X$ is a topology, we have that their union is in the topology aswell. Denote this union as $N := \bigcup_{i\in I} V_i \in \mathcal{O}_X$. Then, we have:

$$\bigcup_{i\in I} S_i = N \cap A, \;\; N \in \mathcal{O}_X.$$

This is precisely the definition of being in the subspace topology, hence the arbitrary union of open sets is open, as desired, which finishes the proof.

□

Proposition 2.2. *Let $(X, \mathcal{O}_X)$ be a topological space and let $A \subseteq X$ be equipped with the subspace topology $\mathcal{O}_A$. Then $S \subseteq A$ is closed in $(A, \mathcal{O}_A)$ iff $S = K \cap A$ for some K closed in $(X, \mathcal{O}_X)$.*

Proof. "$\Rightarrow$" Let S be closed in $(A, \mathcal{O}_A)$. This implies that:

$$A \setminus S \in \mathcal{O}_A.$$

By the definition of subspace topology:

$$A \setminus S \in \mathcal{O}_A \implies \exists U \in \mathcal{O}_X : A \setminus S = U \cap A.$$

We now have to write $S = K \cap A$ for some K closed in $(X, \mathcal{O}_X)$. We know that $X \setminus U$ is closed in $(X, \mathcal{O}_X)$, since U was open. We claim choosing $K = X \setminus U$ does the job. This can be seen as follows:

$$(X \setminus U) \cap A = (X \cap A) \setminus (U \cap A) = A \setminus (U \cap A).$$

Now using the assumption that S is closed in $\mathcal{O}_A$:

$$A \setminus (U \cap A) = A \setminus (A \setminus S) = S.$$

Thus, indeed, we have that:

$$S = (X \setminus U) \cap A.$$

Hence, if S is closed in $(A, \mathcal{O}_A)$, it can be written as $S = K \cap A$ for some K closed in $(X, \mathcal{O}_X)$, namely $K = X \setminus U$.

"$\Leftarrow$" Let $S = K \cap A$ for some K closed in $(X, \mathcal{O}_X)$. We have to show that $K \cap A$ is closed in $(A, \mathcal{O}_A)$:

$$K \cap A \text{ is closed in } (A, \mathcal{O}_A) \iff A \setminus (K \cap A) \in \mathcal{O}_A \iff \exists U \in \mathcal{O}_X : A \setminus (K \cap A) = U \cap A.$$

Thus, we only have to show that there exists $U \in \mathcal{O}_X : A \setminus (K \cap A) = U \cap A$. But this indeed holds if we let $U = X \setminus K$, since:

1. $U = X \setminus K$ is open in $(X, \mathcal{O}_X$, since K is closed by assumption;
2. $(X \setminus K) \cap A = (X \cap A) \setminus (K \cap A) = A \setminus (K \cap A)$, as desired.

$\square$

Proposition 2.3. *Let $(X, \mathcal{O}_X)$ be a topological space and let $A \subseteq X$ be open in $(X, \mathcal{O}_X)$. Then :*

$$U \subseteq A \in \mathcal{O}_A \iff U \in \mathcal{O}_X.$$

Mnemonic: A subspace of a topological space is open iff it is open with respect to the subspace topology.

Proof. "$\Rightarrow$" Let $U \subseteq A \in \mathcal{O}_A$. This implies that:

$$\exists V \in \mathcal{O}_X : U = V \cap A.$$

But since $A \in \mathcal{O}_X$ and $\mathcal{O}_X$ is a topology, we have that $V \cap A \in \mathcal{O}_X$, thus $U \in \mathcal{O}_X$, as desired.
$\Leftarrow$ Let $U \subseteq A \in \mathcal{O}_X$. But since U is a subset of A, we have that $U = U \cap A$. Thus, there exists $V \in \mathcal{O}_X$, such that $U = V \cap A$, namely $V = U$. This shows that $U \in \mathcal{O}_A$. $\square$

Lemma 2.4. *Let $(X, \mathcal{O}_X)$ be a topological space and $(A_i)_{i \in I}$ be subsets of X equipped with the subspace topology. Then the inclusion maps*

$$j_i : A_i \to X, \;\; j_i(U) = U$$

are continuous.

Proof. j_i are continuous $\iff (j_i)^{-1}(U) \in \mathcal{O}_A \; \forall U \in \mathcal{O}_X$. Now let $U \in \mathcal{O}_X$. Then:

$$(j_i)^{-1}(U) = \{U \in A_i | j_i(U) \in X\} = \{U \in A_i | U \in X\} = U \cap X.$$

But since $U \in \mathcal{O}_X$, we have that $U \cap X \in \mathcal{O}_A$, by the definition of the subspace topology. Thus, we showed that:

$$(j_i)^{-1}(U) = U \cap X \in \mathcal{O}_X, \;\; \forall U \in \mathcal{O}_X,$$

as desired. $\square$

We now come to an important theorem related to characterization of topologies via closed sets.

Theorem 2.5. *Let X be a set and $\mathcal{O}_X \subseteq P(X)$. Then $\mathcal{O}_X$ is a topology on X iff*

1. *$X, \varnothing$ are closed sets in $\mathcal{O}_X$;*
2. *The union of finitely many closed sets is closed in $\mathcal{O}_X$;*
3. *The intersection of arbitrarily many closed sets is closed in $\mathcal{O}_X$,*

where a closed set V in $\mathcal{O}_X$ is defined as a subset of X, such that $X \setminus V \in \mathcal{O}_X$.

Proof. "$\Rightarrow$ Assume $\mathcal{O}_X$ is a topology on X. Then:

1. X is closed in $\mathcal{O}_X$, since $X \setminus X = \varnothing \in \mathcal{O}_X$. Similarly, $\varnothing$ is closed in $\mathcal{O}_X$, since $X \setminus \varnothing = X \in \mathcal{O}_X$.

2. Let $(V_i)_{i \in \mathbb{N}}$ be a finite collection of closed sets in $\mathcal{O}_X$. Then, we have:

$$(X \setminus V_i)_{i \in \mathbb{N}} \in \mathcal{O}_X.$$

   But since $\mathcal{O}_X$ was supposed to be a topology, the intersection of finitely many open sets is still open in the topology:

$$\bigcap_{i=1}^{n} (X \setminus V_i) \in \mathcal{O}_X.$$

   By the De Morgan laws A.2 we can rewrite this as:

$$\bigcap_{i=1}^{n} (X \setminus V_i) = X \setminus \left(\bigcup_{i=1}^{n} V_i \right).$$

   Since the two sets are equal and the LHS is open in $\mathcal{O}_X$, so is the RHS.
   More explicitly: $\left(X \setminus \left(\bigcup_{i=1}^{n} V_i \right) \right) \in \mathcal{O}_X$. Hence, $\bigcup_{i=1}^{n} V_i$ is closed in $\mathcal{O}_X$.

3. Let $(V_i)_{i \in I}$ be an arbitrary collection of closed sets in $\mathcal{O}_X$. Then, we have:

$$(X \setminus V_i)_{i \in I} \in \mathcal{O}_X.$$

   But since $\mathcal{O}_X$ was supposed to be a topology, the union of arbitrarily many open sets is still open in the topology:

$$\bigcup_{i \in I} (X \setminus V_i) \in \mathcal{O}_X.$$

   By the De Morgan laws A.1 we can rewrite this as:

$$\bigcup_{i \in I} (X \setminus V_i) = X \setminus \left(\bigcap_{i \in I} V_i \right)$$

   Since the two sets are equal and the LHS is open in $\mathcal{O}_X$, so is the RHS.
   More explicitly: $\left(X \setminus \left(\bigcap_{i \in I} V_i \right) \right) \in \mathcal{O}_X$. Hence $\bigcap_{i \in I} V_i$ is closed in $\mathcal{O}_X$.

"$\Leftarrow$" Conversely, suppose the axioms $1-3$ hold for a collection of subsets $\mathcal{O}_X$ of the power set $P(X)$. Then we have a collection of subsets, such that:

1. X is closed in $\mathcal{O}_X$. This implies $X \setminus X = \varnothing \in \mathcal{O}_X$. Similarly, $\varnothing$ is closed in $\mathcal{O}_X$, that is $X \setminus \varnothing = X \in \mathcal{O}_X$.

2. If $(V_i)_{i\in\mathbb{N}}$ are closed in $\mathcal{O}_X$, then $\bigcup\limits_{i=1}^{n} V_i$ is closed in $\mathcal{O}_X$. This can be rewritten as:

$$(X \setminus V_i)_{i\in\mathbb{N}} \in \mathcal{O}_X \implies \left(X \setminus \left(\bigcup_{i=1}^{n} V_i\right)\right) \in \mathcal{O}_X.$$

   Using the De Morgan laws A.1, we can rewrite the right hand side of the implication to obtain:

$$(X \setminus V_i)_{i\in\mathbb{N}} \in \mathcal{O}_X \implies \bigcap_{i=1}^{n} (X \setminus V_i) \in \mathcal{O}_X.$$

3. If $(V_i)_{i\in I}$ are closed in $\mathcal{O}_X$, then $\bigcap\limits_{i\in I} V_i$ is closed in $\mathcal{O}_X$. This can be rewritten as:

$$(X \setminus V_i)_{i\in I} \in \mathcal{O}_X \implies \left(X \setminus \left(\bigcap_{i\in I} V_i\right)\right) \in \mathcal{O}_X.$$

   Using the De Morgan laws A.1, we can rewrite the right hand side of the implication to obtain:

$$(X \setminus V_i)_{i\in\mathbb{N}} \in \mathcal{O}_X \implies \bigcup_{i\in I} (X \setminus V_i) \in \mathcal{O}_X.$$

Thus, we can see that the collection

$$A := \{(X \setminus V_i)_{i\in I} | V_i \quad \text{is closed in} \quad \mathcal{O}_X\}$$

satisfies the axioms of a topology. □

Corollary 2.6. *Let $(X, \mathcal{O}_X)$ be a topological space. Then:*

1. *finite unions of closed sets are closed;*

2. *arbitrary intersections of closed sets are closed.*

Remark 2.7. *A map is continuous iff the preimages of closed sets are closed. Thus, if we have a closed cover, the proof of theorem 2.6. in the lecture works out the same way, except that the closed cover needs be finite, since arbitrary union of closed sets is not guaranteed to be closed anymore, as we saw in the above proof.*

Proposition 2.8. *Let $(X, \mathcal{O}_X)$ be a topological space and let $f : Y \to X$ be a set map. Then, the*

induced topology on Y by f defined as

$$\mathcal{O}_f := \{f^{-1}(U) \subseteq Y | U \in \mathcal{O}_X\}$$

is a topology on Y.

Proof. We show that it is a topology by proving that it satisfies all the three axioms.

1. $f^{-1}(X) = \{y \in Y | f(y) \in X\} = Y$, $f^{-1}(\varnothing) = \varnothing$. Thus, we have that $Y, \varnothing \in \mathcal{O}_f$.

2. Let $U, V \in \mathcal{O}_f$. Then, we have: $\exists A \in \mathcal{O}_X : f^{-1}(A) = U$, $\exists B \in \mathcal{O}_X : f^{-1}(B) = V$. Thus:

$$U \cap V = f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B).$$

   But since $A, B \in \mathcal{O}_X$ and $\mathcal{O}_X$ is a topology, we have that $A \cap B \in \mathcal{O}_X$. Denote $\rho := A \cap B$. This leads to:

$$\exists \rho \in \mathcal{O}_X : U \cap V = f^{-1}(\rho).$$

   Hence, $U \cap V \in \mathcal{O}_f$.

3. Let $(V_i)_{i \in I} \in \mathcal{O}_f$. Then, we have : $\exists (A_i)_{i \in I} \in \mathcal{O}_X : f^{-1}(A_i) = V_i$. Thus:

$$\bigcup_{i \in I} V_i = \bigcup_{i \in I} f^{-1}(A_i) = f^{-1}\left(\bigcup_{i \in I} A_i\right).$$

   But since $(A_i)_{i \in I} \in \mathcal{O}_X$ and $\mathcal{O}_X$ is a topology, we have that $\bigcup\limits_{i \in I} A_i \in \mathcal{O}_X$. Denote $\alpha := \bigcup\limits_{i \in I} A_i$. This leads to:

$$\exists \alpha \in \mathcal{O}_X : \bigcup_{i \in I} V_i = f^{-1}(\alpha).$$

   Hence, $\bigcup\limits_{i \in I} V_i \in \mathcal{O}_f$.

□

2.2 Embeddings

Proposition 2.9. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an embedding iff it induces a map $\tilde{f} : X \to f(X) : \tilde{f}(x) = f(x)$, which is a homeomorphism, given $f(X)$ is equipped with the subspace topology of $\mathcal{O}_Y$.*

Proof. "⇒" Let f be an embedding. Then f is continuous and injective. Thus $\tilde{f} : X \to f(X), \tilde{f}(a) := f(a)$ is continuous and bijective. We have to show that $\tilde{f}$ is open, since we know that a continuous bijective open map is a homeomorphism. Here $f(X)$ has the subspace topology $\mathcal{O}_{f(X)}$ inherited from $(Y, \mathcal{O}_Y)$.
Let $V \in \mathcal{O}_X$. Since f is am embedding, $O_x = O_f$, thus:

$$V \in \mathcal{O}_X \iff \exists S \in \mathcal{O}_Y : V = f^{-1}(S).$$

We now show that $f^{-1}(S) = \tilde{f}^{-1}(S \cap f(X))$. This can be seen from: $y \in f^{-1}(S) \iff f(y) \in S$.

But $f(y) \in f(X)$ so $y \in f^{-1}(S) \iff f(y) \in S \cap f(X) \iff \tilde{f}(y) \in S \cap f(X) \iff y \in \tilde{f}^{-1}(S \cap f(X))$.

Hence, we have that:

$$V \in \mathcal{O}_X \iff \exists S \in \mathcal{O}_Y : V = \tilde{f}^{-1}(S \cap f(X)).$$

Now, to show $\tilde{f}$ is open, consider the image of V.

$$\tilde{f}(V) = \tilde{f}\left(\tilde{f}^{-1}(S \cap f(X))\right) = \left(\tilde{f} \circ \tilde{f}^{-1}\right)(S \cap f(X)) = S \cap f(X).$$

But since $S \in \mathcal{O}_Y$, this is precisely the definition of an open set in the subspace topology $\mathcal{O}_{f(X)}$:

$$\exists S \in \mathcal{O}_Y : \tilde{f}(V) = S \cap f(X).$$

Thus, $\tilde{f}$ maps open sets to open sets, i.e. is open. This shows that $\tilde{f}$ is a homeomorphism.
"$\Leftarrow$" Conversely, suppose $\tilde{f} : X \to \tilde{Y}$ is a homeomorphism, where $\tilde{Y} \subseteq Y$. Now using lemma B.1, we can define an injective function:

$$f : X \to Y, f(x) := \tilde{f}(x).$$

Let $\mathcal{O}_X$ be a topology on X and $U \in \mathcal{O}_X$. We want to show that $\mathcal{O}_X = \mathcal{O}_f$. Since $\tilde{f}$ is a homeomorphism, we have that $\tilde{f}(U)$ is open in $\tilde{Y}$. This is equivalent to saying:

$$\exists V \in \mathcal{O}_Y : \tilde{f}(U) = V \cap \tilde{Y}.$$

Thus, we have:

$$f^{-1}(V) = \tilde{f}^{-1}(V) = \tilde{f}^{-1}(\tilde{f}(U)) = U,$$

so all open $U \in \mathcal{O}_X$ are of the form $f^{-1}(V)$ for some $V \in \mathcal{O}_Y$. This shows $\mathcal{O}_X = \mathcal{O}_f$, as desired. $\square$

2.3 Products and initial topology

Here we will take a slightly different approach from the lecture, however equivalent.

Definition 2.10. *Given a set X and a family $(X_i)_{i \in I}$ of topological spaces with functions*

$$f_i : X \to X_i,$$

the initial topology τ on X is the coarsest topology on X such that all $f_i : (X, \tau) \to X_i$ are continuous.

It is nice to define it as such, since one can see special cases arising as examples:

1. subspace topology: if we choose for f_i the inclusion map;
2. product topology: if we choose for $f_i = \mathrm{pr}_i$ the projection maps.

Now let us show that our definition is equivalent to the one given in the lecture:

Proposition 2.11. *Let X be any set together with functions $f_i : X \to X_i$ taking values in topological spaces $(X_i)_{i\in I}$, and suppose the topology on X is the coarsest topology making all f_i continuous. Then the collection*

$$S := \{f_i^{-1}(U_i)|i \in I, U_i \subseteq X_i \text{ open}\}$$

is a subbasis for the topology on X.

Proof. For all f_i to be continuous, we must have that $f_i^{-1}(U_i)$ has to be open in X for each open $U_i \in X_i$. These sets define a subbasis

$$S := \{f_i^{-1}(U_i)|i \in I, U_i \subseteq X_i \text{ open}\}$$

for some topology τ: the open sets in this topology are precisely the unions of finite intersections of sets of the form $f_i^{-1}(U_i)$. However, this is the coarsest topology. To see this, consider any other topology τ', which contains S (this is the minimality condition needed for f_i to be continuous). Then using the equivalent description of subbasis from Wikipedia:

$$S \subset \tau' \implies \tau \subseteq \tau'.$$

This shows that τ is indeed the coarsest such topology. □

Remark 2.12. *To see the equivalence from Wikipedia a bit more explicitly, consider the following: Let S be a subbasis of τ. Then $\mathcal{B} := \{$finite intersections of elements of $S\}$ is a basis for τ. Let τ' be another topology, which containts S. Then, by the definition of a topology, τ' contains all finite intersections of elements of S, that is $\mathcal{B} \in \tau'$. Since $\mathcal{B}$ is a basis for τ, we have that every open set in τ is a union of elements in $\mathcal{B}$. Moreover, all unions of elements of $\mathcal{B}$ are in τ' by the definition of topology. Thus, we conclude:*

$$A \in \tau \implies A \in \tau' \iff \tau \subseteq \tau'.$$

Corollary 2.13. *A subbasis of the product topology is given by cylinders:*

$$S = \{pr_i^{-1}(U_i)|i \in I, U_i \subseteq X_i \text{ open}\}.$$

Theorem 2.14. *Let Z be a topological space and $(X_i)_{i\in I}$ a family a topological spaces. Equip X with the product topology. Then, a function $f : Z \to X$ is continuous iff each $f_i : Z \to X_i, f_i := \mathrm{pr}_i \circ f$ is continuous.*

Proof. "$\Rightarrow$" Suppose f is continuous. We know that pr_i are continuous from the lecture. Thus, $f_i = \mathrm{pr}_i \circ f$ is continuous because the composition of continuous maps is continuous.
"$\Leftarrow$" Suppose each f_i are continuous. Using the remark from the lecture, which was proven in the exercises, that a map is continuous iff preimages of elements of a subbasis are open in the domain,

it is enough to check preimages of cylinder sets. So, suppose V is a cylinder set in X. This can be written as:

$$V \in S \implies V = pr_i^{-1}(U_i) : i \in I, U_i \subseteq X_i \text{ open.}$$

Checking continuity on the cylinder sets gives:

$$f^{-1}(V) = f^{-1}\left(\mathrm{pr}_i^{-1}(U_i)\right) = (\mathrm{pr}_i \circ f)^{-1}(U_i) = f_i^{-1}(U_i).$$

However, since U_i are open in X_i and f_i are continuous by assumption, we obtain that f^{-1} is continuous, as desired.

□

2.3.1 Physicist approach: product of two topological spaces

Although we have discussed the infinite product case in the lecture, and have proved above the universal property for infinite products, let us work out the finite case explicitly aswell, as exercise.

Definition 2.15. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. The product of X and Y is the topological space $(X \times Y, \mathcal{O}_{X\times Y})$, where:*

$$\mathcal{O}_{X\times Y} := \left\{ \bigcup_{i\in I} P_i | P_i \in \mathcal{O}_X \times \mathcal{O}_Y \right\}$$

is called the product topology on $X \times Y$.

Definition 2.16. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. The set $\mathcal{O}_{X\times Y}$ defined implicitly by*

$$U \in \mathcal{O}_{X\times Y} \iff \forall p \in U : \exists (S,T) \in \mathcal{O}_X \times \mathcal{O}_Y : S \times T \subseteq U,\ p \in S \times T$$

is called the product topology on $X \times Y$.

Remark 2.17. *The last part in the definition that $p \in S \times T$ is crucial, because otherwise literally any set would be open: for any p we could choose $S = T = \varnothing$.*

Proposition 2.18. *The definitions 2.15 and 2.16 are equivalent.*

Proof. $\Rightarrow$ Accepting the first definition, the open sets $U \in \mathcal{O}_{X\times Y}$ are of the form: $U = \bigcup_{i\in I} P_i$ for $P_i \in \mathcal{O}_X \times \mathcal{O}_Y$. Now, let $p \in U$. Using the definition of union, this means, that $p \in P_i$ for some $i \in I$. Moreover, $P_i \subseteq U$. But since $P_i \in \mathcal{O}_X \times \mathcal{O}_Y$, we have that $P_i = S \times T$ for some $S \in \mathcal{O}_X$ and some $T \in \mathcal{O}_Y$. But this is exactly what definition 2.16 says. We can do this for any p, since it was arbitrary.
$\Leftarrow$ Accepting the second definition, we have that every element p is contained in some $S \times T \subseteq U$, for S open in X, T open in Y. If we take such $S \times T$ for every p, we get that U is a union of sets of the form $S \times T$, as desired. □

We still have to show that what we defined above indeed is a topological space. To this end, we

have to verify that $\mathcal{O}_{X\times Y}$ satisfies the axioms of a topology.

Proof. 1. (a) $(\varnothing,\varnothing)\in\mathcal{O}_{X\times Y}$, since:

i. $\varnothing\in\mathcal{O}_X$, since $(X,\mathcal{O}_X)$ is a topological space;

ii. $\varnothing\in\mathcal{O}_Y$, since $(Y,\mathcal{O}_Y)$ is a topological space.

(b) $X\times Y\in\mathcal{O}_{X\times Y}$, since:

i. $X\in\mathcal{O}_X$, since $(X,\mathcal{O}_X)$ is a topological space;

ii. $Y\in\mathcal{O}_Y$, since $(Y,\mathcal{O}_Y)$ is a topological space.

2. Let $(U_i)_{i\in I}$ be open in $X\times Y$, i.e. $(U_i)_{i\in I}\in\mathcal{O}_{X\times Y}$. We have to show that $U=\bigcup\limits_{i\in I}U_i\in\mathcal{O}_{X\times Y}$. To this end, take $p\in U$. Then, by definition of U, we have that $p\in U_i$ for some $i\in I$. But we know that U_i for fixed $i\in I$ is open, from the assumption. Using the definition of product topology:

$$U_i\in\mathcal{O}_{X\times Y}\iff\forall p\in U_i:\exists(S,T)\in\mathcal{O}_X\times\mathcal{O}_Y:S\times T\subseteq U_i,p\in S\times T.$$

But since $U_i\subseteq U$, we also immediately have that $S\times T\subseteq U$. This shows that U is open in $X\times Y$, according to definition 2.16.

3. Let U_1,U_2 be open in $X\times Y$, i.e. $U_1,U_2\in\mathcal{O}_{X\times Y}$. We have to show that $U_1\cap U_2$ is also open in $X\times Y$. Let $p\in U_1\cap U_2$. This means that $p\in U_1$ and $p\in U_2$. Using that U_1 is open in $X\times Y$:

$$\forall p\in U_1:\exists(S,T)\in\mathcal{O}_X\times\mathcal{O}_Y:S\times T\subseteq U_1,p\in S\times T.$$

Using that U_2 is open in $X\times Y$:

$$\forall p\in U_2:\exists(S',T')\in\mathcal{O}_X\times\mathcal{O}_Y:S'\times T'\subseteq U_2,p\in S'\times T'.$$

Therefore, we have for all $p\in U_1\cap U_2$ the following:

$$p\in(S\times T)\cap(S'\times T')\iff p\in\underbrace{(S\cap S')}_{:=W}\times\underbrace{(T\cap T')}_{:=Q},$$

Since $\mathcal{O}_X$ is a topology, we know that W is open in X and since $\mathcal{O}_Y$ is a topology, we know that Q is open in Y. Therefore, for each $p\in U_1\cap U_2$ we have that:

$$\forall p\in U_1\cap U_2:\exists(W,Q)\in\mathcal{O}_X\times\mathcal{O}_Y:W\times Q\subseteq U_1\cap U_2,p\in W\times Q,$$

which proves that $U_1\cap U_2$ is open in $X\times Y$ according to definition 2.16.

Therefore, we showed that the product topology satisfies all the axioms of a topology, hence deserves its name. □

We will need a following proposition to prove the universal property for finite case.

Proposition 2.19. *Let A be a set and τ, σ be two different topologies on the same set. If the identity map $(A, \sigma) \to (A, \tau)$ is a homeomorphism, then $\sigma = \tau$ as subsets of $\mathcal{P}(A)$.*

Proof. Let $U \in \tau$. Since the identity map is continuous, we have that the preimage of U under the identity is open in σ. That is, $\mathrm{id}^{-1}(U) = U \in \sigma$. Therefore:

$$U \in \tau \implies U \in \sigma. \tag{1}$$

Moreover, since the identity is a homeomorphism, its inverse is also continuous. Therefore, suppose $U \in \sigma$. Then, the preimage under its inverse is again U, which is open in τ, since the identity is the homeomorphism. Therefore:

$$U \in \sigma \implies U \in \tau. \tag{2}$$

Using 1,2 we conclude that $\sigma = \tau$. $\square$

So far, we have established, that we can equip the cartesian product with a topology called the product topology. Now we come to the point of asking, whether this topology satisfies some universal property and whether it is unique. It will turn out that the answer is positive. Before that, let us review something about cartesian products.

Proposition 2.20. *Let X, Y, Z be sets. Giving a function $f : Z \to X \times Y$ is equivalent to the data of providing a function $f_1 : Z \to X$ and a function $f_2 : Z \to Y$. In set theoretic terms, there exists a bijection:*

$$\{f : Z \to X \times Y\} \cong \{(f_1 : Z \to X, f_2 : Z \to Y)\}$$

between the set of functions from Z to $X \times Y$ and the set of pairs of functions $fromZ$ to X and from Z to Y.

Proof. To prove the claim, we only need to provide the isomorphism between the two sets. To this end, let us denote:

$$S_1 := \{f : Z \to X \times Y\}, S_2 := \{(f_1 : Z \to X, f_2 : Z \to Y)\}.$$

Now, to investigate this a bit, we have to see how we get from S_1 to S_2. Suppose we have a function $f \in S_1$,i.e. $f : Z \to X \times Y$. We can obtain functions $f_1 : Z \to X$ and $f_2 : Z \to Y$ as follows:

$$f_1 := \mathrm{pr}_1 \circ f, f_2 := \mathrm{pr}_2 \circ f.$$

On the other hand, given $f_1 : Z \to X, f_2 : Z \to Y$, we can obtain a map $f : Z \to X \times Y$ as follows:

$$f : Z \to X \times Y, f(z) := (f_1(z), f_2(z)).$$

Thus, we have our two candidates:

$$\psi : S_1 \to S_2, \psi(f) := (\mathrm{pr}_1 \circ f, \mathrm{pr}_2 \circ f);$$

$$\phi : S_2 \to S_1, \phi(f_1, f_2) := f.$$

They do the job, because they are inverses of each other:

$$(\psi \circ \phi)(f_1, f_2) = \psi(\phi(f_1, f_2)) = \psi(f) = (\mathrm{pr}_1 \circ f, \mathrm{pr}_2 \circ f) = (f_1, f_2) \implies \psi \circ \phi = \mathrm{id}.$$

$$(\phi \circ \psi)(f) = \phi(\psi(f)) = \phi(\mathrm{pr}_1 \circ f, \mathrm{pr}_2 \circ f) = \phi(f_1, f_2) = f \implies \phi \circ \psi = \mathrm{id}.$$

□

Now, we are ready to ready to state the universal property.

Theorem 2.21. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then, there exists a topology on $X \times Y$, such that:*

1. *The projection maps $pr_1 : X \times Y \to X$, $pr_2 : X \times Y \to Y$ are continuous;*
2. *For any topological space $(Z, \mathcal{O}_Z)$ and for any pair of functions $f_1 : Z \to X, f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is continuous if and only if f_1, f_2 are.*

This can be expressed also as follows: Let be the product space X together with the canonical projections. Then if Z is a topological space and for every $i \in I, f_i : Z \to X_i$ is a continuous map, then there exists precisely one continuous map $f : Z \to X$ such that for each $i \in I$ the following diagram commutes:

$$\begin{tikzcd} & X \arrow[d, "pr_i"] \\ Z \arrow[ur, "f"] \arrow[r, "f_i"'] & X_i. \end{tikzcd}$$

The fact that this diagram commutes means that $f_i = pr_i \circ f$. In particular, f_i are continuous iff f is.

Proof. To show that there exists a topology on $X \times Y$, which fulflls the theorem, it is enough to give one. We proved that the product topology is a topology. We now show that it satisfies the properties given in the theorem.

1. pr_1 is continuous, since for any open subset $U \in \mathcal{O}_X$, we have that:

   $$\mathrm{pr}_1^{-1}(U) = U \times Y.$$

   Since we assumed that $U \in \mathcal{O}_X$ and since $Y \in \mathcal{O}_Y$, because $\mathcal{O}_Y$ is a topology, we have that $U \times Y \in \mathcal{O}_{X \times Y}$, as desired. Similarly, pr_2 is continuous, since for any open subset $V \in \mathcal{O}_Y$, we have that:

   $$\mathrm{pr}_2^{-1}(V) = X \times V.$$

   As $\mathcal{O}_X$ is a topology, $X \in \mathcal{O}_X$ and $V \in \mathcal{O}_Y$. Therefore, $X \times V \in \mathcal{O}_{X \times Y}$ as desired. This shows that

both pr_1 and pr_2 are continuous.

2. "$\Rightarrow$" Suppose f is continuous. Then, for every open set $W \in \mathcal{O}_{X\times Y}$, we have that the preimage $f^{-1}(W)$ is an open set of Z. We want to show that f_1 and f_2 are both continuous, that is, for every open set $U \in \mathcal{O}_X$ and $V \in \mathcal{O}_Y$, the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open subsets of Z. To this end, let $U \in \mathcal{O}_X$ and $V \in \mathcal{O}_Y$ arbitrary. We look at their preimages under f_1, f_2:

$$f_1^{-1}(U) = (\mathrm{pr}_1 \circ f)^{-1}(U) = \left(f^{-1} \circ \mathrm{pr}_1^{-1}\right)(U) = f^{-1}\left(\left(\mathrm{pr}_1^{-1}\right)(U)\right) = f^{-1}(U \times Y). \tag{3}$$

$$f_2^{-1}(V) = (\mathrm{pr}_2 \circ f)^{-1}(V) = \left(f^{-1} \circ \mathrm{pr}_2^{-1}\right)(V) = f^{-1}\left(\left(\mathrm{pr}_2^{-1}\right)(V)\right) = f^{-1}(X \times V). \tag{4}$$

Both $U \times Y$ and $X \times V$ are open in the product topology, as we have argued above. Since f is continuous, the right hand sides of 3, 4 are open, because the preimage of every open set in the target is open in the domain. Then, by equating the left hand side with the right hand side, we get that $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in Z. That is, both f_1 and f_2 are continuous, as desired.
"$\Leftarrow$" Suppose both f_1 and f_2 are continuous. We want to show that f is also continuous. For all open sets $U \in \mathcal{O}_X, V \in \mathcal{O}_Y$, we have $f_1^{-1}(U) \in \mathcal{O}_Z, f_2^{-1}(V) \in \mathcal{O}_Z$ by continuity. Once again, we have:

$$f_1^{-1}(U) = (\mathrm{pr}_1 \circ f)^{-}1(U) = \left(f^{-1} \circ \mathrm{pr}_1^{-1}\right)(U) = f^{-1}\left(\left(\mathrm{pr}_1^{-1}\right)(U)\right) = f^{-1}(U \times Y). \tag{5}$$

$$f_2^{-1}(V) = (\mathrm{pr}_2 \circ f)^{-1}(V) = \left(f^{-1} \circ \mathrm{pr}_2^{-1}\right)(V) = f^{-1}\left(\left(\mathrm{pr}_2^{-1}\right)(V)\right) = f^{-1}(X \times V). \tag{6}$$

The left hand side of both 5 6 are open, by assumption, thus, by equality, the right hand side is open too. By taking the intersection, we get:

$$f_1^{-1}(U) \cap f_2^{-1}(V) = f^{-1}(U \times Y) \cap f^{-1}(X \times V) = f^{-1}\left((U \times Y) \cap (X \times V)\right) = f^{-1}(U \times V).$$

On the left hand side, we have the intersection of two open sets, which, by the definition of topology, is open. On the right hand side, we have the preimage of $U \times V$ under f. $U \times V$ is an element in the basis $\mathcal{B}$, which generates the product topology and by the equality, $f^{-1}(U \times V)$ is open. This show that the preimage of every basis elment of $X \times Y$ is open.
Now, consider an arbitrary open subset $W \in \mathcal{O}_{X\times Y}$. Since $U \times V$ is a basis for the product topology, W can be written as a union of basis elements, that is:

$$W = \bigcup_{i\in I}(U_i \times V_i).$$

Consider the preimage of W under f:

$$f^{-1}(W) = f^{-1}\left(\bigcup_{i\in I}(U_i \times V_i)\right) = \bigcup_{i\in I} f^{-1}(U_i \times V_i). \tag{7}$$

Now, the right hand side of 7 is an arbitrary union of open sets, since we showed that $f^{-1}(U_i \times V_i)$ are open. An arbitrary union of open sets is open, which shows that the right hand side is open.

By equality, the left hand side is open aswell. Therefore, the preimage of every open set W in $X \times Y$ is open in Z, which concludes the continuity of f.

$\square$

This shows, that the product topology satisfies the universal property. Now, we come to uniqueness.

Theorem 2.22. *Let $Z = X \times Y$. Let $T_1 = (X \times Y, \mathcal{T}_1)$ and $T_2 = (X \times Y, \mathcal{T}_2)$ be topological spaces satisfying the universal property. Then $\mathcal{T}_1 = \mathcal{T}_2$.*

Before turning to the proof, the above theorem simply means that, if a topology, which satisfies the universal property exists, then it is unique.

Proof. Assume that T_1 satisfies the universal property and take $Z = T_2$. Then, we have that $f_1 : X \times Y \to X$, $f_2 : X \times Y \to Y$ with respect to $\mathcal{T}_2$ and $\mathcal{O}_X, \mathcal{O}_Y$ are continuous iff $f : X \times Y \to X \times Y$ is continuous with respect to $\mathcal{T}_2, \mathcal{T}_1$. But at the same time, we know that T_2 satisfies the universal property. In particular, this means, by the first part of the universal property that $f_1 : X \times Y \to X$ and $f_2 : X \times Y \to Y$ are continuous, with respect to $\mathcal{T}_2, \mathcal{O}_X, \mathcal{O}_Y$. Thus, we can conclude that $f : X \times Y \to X \times Y$ is continuous, with respect to $\mathcal{T}_2, \mathcal{T}_1$.
Now, use the universal property for T_2 and take $Z = T_1$. Then, we have that $f_1 : X \times Y \to X, f_2 : X \times Y \to Y$ are continuous with respect to $\mathcal{T}_1, \mathcal{O}_X, \mathcal{O}_Y$ iff $f : X \times Y \to X \times Y$ is continuous with respect to $\mathcal{T}_1, \mathcal{T}_2$. But since T_1 satisfies the universal property, we know that f_1, f_2 are continuous. Therefore, f is continuous. Therefore, we have a homeomorphism between $T_1 = (X \times Y, \mathcal{T}_1)$ and $T_2 = (X \times Y, \mathcal{T}_2)$, namely the identity map.
Using proposition 2.19, we conclude that $\mathcal{T}_1 = \mathcal{T}_2$. $\square$

Therefore, we have seen that the product topology satisfies the universal property, and we have also seen that it is the unique topology, which satisfies it. Therefore, we are arrive at a third,but equivalent definition of the product topology in terms of the universal property.

Definition 2.23. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. The product topology $\mathcal{O}_{X \times Y}$ on $X \times Y$ is the unique topology, such that:*

1. *The projection maps $pr_1 : X \times Y \to X$, $pr_2 : X \times Y \to Y$ are continuous;*

2. *For any topological space $(Z, \mathcal{O}_Z)$ and for any pair of functions $f_1 : Z \to X, f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is continuous if and only if f_1, f_2 are.*

2.4 Quotient spaces

Proposition 2.24. *Let $(X, \mathcal{O}_X)$ be a topological space and let $\sim$ be an equivalence relation, together with the canonical projection $p : X \to X/\sim$, $x \mapsto [x]_\sim$. Then, the set*

$$\mathcal{O} := \{U \subseteq X/\sim \, | p^{-1}(U) \in \mathcal{O}_X\}$$

is a topology on $X/\sim$.

Proof. We have to verify the three axioms of a topology:

1. $\varnothing \in \mathcal{O}$, since $p^{-1}(\varnothing) = \varnothing \in \mathcal{O}_X$. Similarly, $X/\sim \in \mathcal{O}$, since $p^{-1}(X/\sim) = X \in \mathcal{O}_X$.
2. Let $U_1, U_2 \in \mathcal{O}$. Then $p^{-1}(U_1) \in \mathcal{O}_X$ and $p^{-1}(U_2) \in \mathcal{O}_X$. But we also know that:

$$p^{-1}(U_1) \cap p^{-1}(U_2) = p^{-1}(U_1 \cap U_2).$$

   The LHS is open in $\mathcal{O}_X$, since $\mathcal{O}_X$ is a topology: the intersection of finitely many open sets is still open. Thus, by equality, the RHS is open aswell. This shows that $U_1 \cap U_2 \in \mathcal{O}$.
3. Let $(U_i)_{i \in I} \in \mathcal{O}$. Then $p^{-1}(U_i) \in \mathcal{O}_X \forall i \in I$. But we also know that:

$$\bigcup_{i \in I} p^{-1}(U_i) = p^{-1}\left(\bigcup_{i \in I} U_i\right).$$

   Since the LHS is an arbitrary union of open sets in $(X, \mathcal{O}_X)$, it is open. By equality, so is the RHS. Hence, $\bigcup_{i \in I} U_i$ is open in $\mathcal{O}$, as desired.

□

Proposition 2.25. *The quotient topology $\mathcal{O}$ is the finest topology that makes the canonical projection $p: X \to X/\sim$ continuous.*

Proof. Suppose that $\mathcal{O}'$ is any topology on $X/\sim$, which makes the projection $p: X \to X/\sim$ continuous. Let $U \in \mathcal{O}'$. Then, by continuity of p, we must have that $p^{-1}(U) \in \mathcal{O}_X$. However, by the definition of the quotient topology, we must also have that $U \in \mathcal{O}$. Hence, $\mathcal{O}' \subseteq \mathcal{O}$. This shows, that any topology making the canonical projection p continuous is a subset of the quotient topology. Clearly, the quotient topology makes the canonical projection continuous, so it is the finest one, that does it. □

Remark 2.26. *It is crucial that we say that it makes the canonical projection continuous. The finest topology on $X/\sim$ is the discrete topology. However, this need not make the projection continuous. Note that only if the domain is equipped with the discrete topology, then it makes all maps continuous. This need not be the case for us.*

Proposition 2.27. *Any closed surjection $f: X \to Y$ is a quotient map.*

Proof. "$\Rightarrow$" Clear by continuity: if $V \in \mathcal{O}_Y$, then $f^{-1}(V) \in \mathcal{O}_X$, since f is continuous.
"$\Leftarrow$" Let $V \subseteq Y : f^{-1}(V) \in \mathcal{O}_X$. Then, we have that $X \setminus f^{-1}(V)$ is closed in $\mathcal{O}_X$. Since f is closed, we have that $f(X \setminus f^{-1}(V))$ is closed in O_Y. But we also know that f is surjective. This implies:

$$f \circ f^{-1}(A) = A \ \ \forall A \subseteq Y.$$

We apply this to $A = V$ to obtain:

$$Y \setminus V = f(f^{-1}(Y \setminus V)) = f(X \setminus f^{-1}(V)),$$

where in the last equality we used proposition B.2, which says that preimage of the complement is the complement of the preimage. Hence, we have that:

$$f(X \setminus f^{-1}(V)) = Y \setminus V,$$

which is closed in $\mathcal{O}_Y$. Thus, as desired, $V \in \mathcal{O}_Y$. $\square$

Proposition 2.28. *If a continuous map $f : X \to Y$ between two topological spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ admits local sections, that is: $\forall y \in Y : \exists V_y : y \in V_y, V_y \in \mathcal{O}_y$ and $\exists s_y : V_y \to X : f \circ s_y = id_{V_y}$, then it is a quotient map.*

Proof. Clearly, if we have local sections, we can form an open cover of Y from the collection of open sets from the local sections, as $\bigcup_{y \in Y} V_y \subseteq Y$, because $y \in V_y \implies y \in Y$. Conversely, suppose $y \in Y$. Then, there exists some V_y, so that $y \in V_y$. We conclude: $Y = \bigcup_{y \in Y} V_y$. The map f is clearly surjective. We have to check:

$$\forall y \in Y : \exists x \in X : f(x) = y.$$

This holds, as, for each $y \in Y$, we can find $V_y \in \mathcal{O}_Y : y \in V_y$. Then, $y = (f \circ s_y)(y) = f(s_y(y))$. So, by setting $s_y(y) = x$, we can find for all $y \in Y x \in X : y = f(x)$. We now have to show that $f^{-1}(U) \in \mathcal{O}_X \iff U \in \mathcal{O}_Y$ for f to be a quotient map. The converse direction $\Leftarrow$ is clear:

$$U \in \mathcal{O}_Y \implies f^{-1}(U) \in \mathcal{O}_X, \text{ by continuity.}$$

Let's prove now the direction $\Rightarrow$. Denote:

$$W := f^{-1}(U) \in \mathcal{O}_X.$$

We now want to show that given $W \in \mathcal{O}_X$, it implies that $U \in \mathcal{O}_Y$. This would immediately be true if $U = \bigcup_{y \in Y} s_y^{-1}(W)$, as we know by continuity of s_y that $s_y^{-1}(W) \in \mathcal{O}_Y$ given $W \in \mathcal{O}_X$. Moreover, union of open sets in $\mathcal{O}_Y$ is open. Hence, to conclude the proof, we show that $U = \bigcup_{y \in Y} s_y^{-1}(W)$. To see this:

$$\begin{aligned} z \in \bigcup_{y \in Y} s_y^{-1}(W) &\iff \exists y \in Y : z \in s_y^{-1}(W) \iff \exists y \in Y : z \in V_y \wedge s_y(z) \in W = f^{-1}(U) \\ &\iff \exists y \in Y : z \in V_y \wedge f(s_y(z)) \in U \iff \exists y \in Y : z \in V_y \wedge z \in U \iff z \in U. \end{aligned}$$

We thus have showed that $f : X \to Y$ is a continuous, surjective map, such that $f^{-1}(U) \in \mathcal{O}_X \iff U \in \mathcal{O}_Y$. $\square$

2.4.1 Universal property of topological quotients

We now show that the quotient topology satisfies the universal property of a quotient.

Theorem 2.29. *Let $(X, \mathcal{O}_X)$ be a topological space and let $\sim$ be an equivalence relation. Equip the quotient space with the quotient topology $\mathcal{O}$. Moreover, we define the canonical projection $p : (X, \mathcal{O}_X) \to (X/\sim, \mathcal{O})$. Then, it holds that:*

1. *The fibers of p are equivalence classes of $\sim$, p is surjective.*
2. *p is continuous.*
3. *Let $(Y, \mathcal{O}_Y)$ be a topological space and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map, which is constant on equivalence classes of $\sim$, i.e. $a \sim b \implies f(a) = f(b)$. Then, there exists an induced map $\tilde{f} : (X/\sim, \mathcal{O}) \to (Y, \mathcal{O}_Y)$, $f = \tilde{f} \circ p$. Moreover, f is continuous iff $\tilde{f}$ is.*
4. *The quotient topology $\mathcal{O}$ is the unique topology on $X/\sim$, which satisfies $2, 3$.*

Proof. 1. This was proven in theorem D.9.

2. This follows from $X/\sim$ being equipped with the quotient topology, i.e.:

$$\mathcal{O} = \{U \subseteq X/\sim | p^{-1}(U) \in \mathcal{O}_X\}.$$

Hence, clearly if we pick $U \in \mathcal{O}$, we have $p^{-1}(U) \in \mathcal{O}_X$, which shows continuity.

3. In theorem D.9 it was shown, that $\tilde{f}$ exists and is given by:

$$\tilde{f} : X/\sim \to Y,\ \tilde{f}([x]) = f(x).$$

On the topological level, this is:

$$\tilde{f} : (X/\sim, \mathcal{O}) \to (Y, \mathcal{O}_Y),\ \tilde{f}([x]) = f(x).$$

We now prove that $\tilde{f}$ is continuous iff f is:
"$\Rightarrow$": Suppose $\tilde{f}$ is continuous. Then $f = \tilde{f} \circ p$. Using that p is continuous, as was proven, we have that f is a composition of continuous maps, hence continuous.
"$\Leftarrow$": Suppose f is continuous. $\tilde{f}$ is continuous iff $\forall U \in \mathcal{O}_Y : \tilde{f}^{-1}(U) \in \mathcal{O}$. Let $U \in \mathcal{O}_Y$ be arbitrary. By the definition of the quotient topology $\mathcal{O}$ we have:

$$\tilde{f}^{-1}(U) \in \mathcal{O} \iff p^{-1}\left(\tilde{f}^{-1}(U)\right) \in \mathcal{O}_X.$$

Moreover, using that $\tilde{f} \circ p = f$ leads to:

$$p^{-1}\left(\tilde{f}^{-1}(U)\right) = (\tilde{f} \circ p)^{-1}(U) = f^{-1}(U).$$

By continuity of f, we can conclude that $\tilde{f}^{-1}(U) \in \mathcal{O}_X$, as desired.

4. Suppose there is another topology $\mathcal{O}'$ on $X/\sim$, which satisfies $2,3$. Because the projection has to be continuous according to 2, we have:

$$p : (X, \mathcal{O}_X) \to (X/\sim, \mathcal{O}') \text{ continuous} \iff \forall A \in \mathcal{O}' : p^{-1}(A) \in \mathcal{O}_X.$$

Clearly, by definition of the quotient topology $\mathcal{O}$, we must have that $A \in \mathcal{O}$, as $p^{-1}(A) \in \mathcal{O}_X$. This shows that $\mathcal{O}' \subseteq \mathcal{O}$. On the other hand, $\mathcal{O}'$ has to fulfill 3 too. Since this holds for any topological space $(Y, \mathcal{O}_Y)$, it works in particular for $(X/\sim, \mathcal{O})$. We then have:

$$\tilde{f} : (X/\sim, \mathcal{O}') \to (X/\sim, \mathcal{O}).$$

Then, we have that $f : (X, \mathcal{O}_X) \to (X, \mathcal{O})$ is continuous iff $\tilde{f}$ is. But we know that f is continuous, as the quotient topology makes f continuous. Thus $\tilde{f}$ is continuous. That is:

$$A \in \mathcal{O} \implies A \in \mathcal{O}', \text{ i.e. } \mathcal{O} \subseteq \mathcal{O}'.$$

We showed that $\mathcal{O} \subseteq \mathcal{O}'$ and $\mathcal{O}' \subseteq \mathcal{O}$, thus $\mathcal{O} = \mathcal{O}'$, which finishes the proof.

□

Corollary 2.30. *By theorem D.9 $\tilde{f}$ is unique on the set theoretic level. After having shown that the quotient topology is the unique topology, which satisfies the above universal property, we can conclude that $\tilde{f}$ in point 3 not only exists, but is unique.*

3 Handout3

3.1 The notion of connectedness

In this subsection the proof of the first proposition is not complete: the converse part is missing, as discussed in the Question session, thus let us reprove this.

Proposition 3.1. *A topological space X is connected iff it is not a direct sum of two nonempty spaces $X = X_0 + X_1$.*

Proof. We will use here contrapositivity:
"$\Rightarrow$" connected $\implies X \neq X_0 + X_1 \iff X = X_0 + X_1 \implies$ connected. To this end, let us assume $X = X_0 + X_1$. Then X carries the final topology:

$$U \subseteq X \in \mathcal{O}_X \iff (U \cap X_0 \in \mathcal{O}_{X_0}) \wedge (U \cap X_1 \in \mathcal{O}_{X_1}).$$

Now letting $U = X_0$ and $U = X_1$ respectively:

$$(X_1 \cap X_1 = X_1 \in \mathcal{O}_{X_1}) \wedge (X_0 \cap X_0 = X_0 \in \mathcal{X}_0).$$

Thus X_0 and X_1 are open in X. But since $X = X_0 + X_1$, we have that:

$$X \setminus X_0 = X_1;\ X \setminus X_1 = X_0.$$

Thus both X_0 and X_1 are clopen, which implies that X is not connected.
"$\Leftarrow$" $X \neq X_0 + X_1 \implies$ connected $\iff$ not connected $\implies X = X_0 + X_1$. To this end, assume X is not connected. Then, there exists an open set, which is both open and closed, and is different from the empty set and the whole set. Call this set X_0. Since X_0 is closed, its complement, call it X_1, is open. But at the same time $X \setminus X_1 = X \setminus X \setminus X_0 = X_0$, hence X_1 is also closed. So we have that both X_0 and X_1 are clopen. Thus, $X = X_0 + X_1$. To see this last step, consider:

1. If U is open in X and $X_1, X_2 \subseteq X$, then by the definition of subspace topology, we have that $U \cap X_1 \in \mathcal{O}_{X_1}$ and $U \cap X_2 \in \mathcal{O}_{X_2}$ by the definition of subspace topology.

2. Conversely, if U is a subset of X and X_1, X_2 are open in X such that their union is X, then: $U \cap X_1$ and $U \cap X_2$ are open in the subspace topologies of X_1 and X_2. By opennes of X_1 and X_2 in X,

both intersections $U \cap X_1$ and $U \cap X_2$ are open in X, so the union of these both intersections is also open in X. But this union is again U, because $X_1 \cup X_2 = X$. So U is open in X.

This shows that U is open in X iff U is open in X_1 and U is open in X_2, but this precisely means that $X = X_1 + X_2$ carries the final topology, as desired. □

3.2 Intervals

Proposition 3.2. *Let $f : X \to Y$ be a continuous function and X be connected. Then $im(f) = f(X)$ is connected.*

Proof. We will prove this by contrapositive: we will show that if $f(X)$ is not connected, then X is not connected either.
Suppose $im(f)$ is not connected. Then it can be written as the union of two non-empty, disjoint open sets:

$$f(X) = O_1 \cup O_2, \;\; O_1, O_2 \in \mathcal{O}_Y.$$

Since the map in general is not surjective, we have that $f(X) \subseteq Y$). Thus, $f(X)$ carries the subspace topology. Hence:

$$\exists U_1, U_2 \in \mathcal{O}_{f(X)} : O_1 = f(X) \cap U_1; \;\; O_2 = f(X) \cap U_2.$$

Since $U_1, U_2 \subseteq Y$, we have that $f^{-1}(U_1), f^{-1}(U_2) \subseteq X$. this implies that:

$$f^{-1}(f(X) \cap U_1) = X \cap f^{-1}(U_1) = f^{-1}(U_1); \;\; f^{-1}(f(X) \cap U_2) = X \cap f^{-1}(U_2) = f^{-1}(U_2).$$

Hence, we conclude:

$$f^{-1}(U_1) = f^{-1}(O_1); \;\; f^{-1}(U_2) = f^{-1}(O_2).$$

By continuity of f, $f^{-1}(O_1), f^{-1}(O_2)$ are disjoint open sets in X, which are non-empty and

$$f^{-1}(O_1) \cup f^{-1}(O_2) = X.$$

This shows that X is not connected, as desired. □

Corollary 3.3. *Connectedness is preserved under homeomorphisms. Hence, it is a topological property, which will be formally introduced in the next chapter.*

Proposition 3.4. *Let $(X, \mathcal{O}_X)$ be a connected topological space and $Y \subseteq X$ be a subspace equipped with the subspace topology*

$$\mathcal{O}_Y := \{Y \cap U | U \in \mathcal{O}_X\}.$$

Then $(Y, \mathcal{O}_Y)$ need not be connected.

Proof. It is enough to find an example, where the subspace is not connected. To this end, let our

space be:

$$X := \{-1, 0, 1\}.$$

We equip it with the following topology:

$$\mathcal{O}_X := \{\varnothing, \{-1\}, \{1\}, \{-1, 1\}, \{-1, 0, 1\}\}.$$

Then the closed sets are:

$$\{-1, 0, 1\}, \{0, 1\}, \{-1, 0\}, \{0\}, \varnothing.$$

Clearly, the only clopen sets are $X, \varnothing$, hence $(X, \mathcal{O}_X)$ is connected. Consider the subset $Y := \{-1, 1\}$. If we equip it with the subspace topology, we get:

$$\mathcal{O}_Y = \{\varnothing, \{-1\}, \{1\}, \{-1, 1\}\},$$

i.e. it is the discrete topology. In particular, $\{-1\}, \{1\}$ are both open and closed in $\mathcal{O}_Y$, hence $(Y, \mathcal{O}_Y)$ is not connected. □

3.3 Connected components

Definition 3.5. *Let x, y be two points in a topological space $(X, \mathcal{O}_X)$. They are called* equivalent *if there exists a connected subset $A \subseteq X : x \in A, y \in A$.*

Lemma 3.6. *Being equivalent is an equivalence relation on any topological space $(X, \mathcal{O}_X)$*

Proof. We have to check that it is symmetric,reflexive, transitive:

1. symmetric: $\forall x, y \in X : x \sim y \implies y \sim x$. This is clear, since $x \sim y \implies \exists A$ connected: $x \in A, y \in A$. But if $x \in A$ and $y \in A$, then aswell $y \in A, x \in A$, thus $y \sim x$.

2. reflexive: $\forall x \in X : x \sim x$. This holds, since $A = \{x\}$ is connected, as it is a singleton.

3. transitive: $\forall x, y, z \in X : x \sim y$ and $y \sim z \implies x \sim z$. Suppose $x \sim y$ and $y \sim z$. We then have two connected sets:

$$\exists A \subseteq X : x \in A, y \in A, \text{ and } \exists B \subseteq X : y \in B, z \in B.$$

Since the intersection $A \cap B$ is non-empty, using the theorem from the lecture, we can conclude that $A \cup B$ is connected. But $A \cup B$ contains both x and z, hence $x \sim z$, as desired.

□

Remark 3.7. *The statement "being equivalent is an equivalence relation" is said in the lecture otherwise, namely: "two points lie in the same connected component if and only if both are contained in some connected set". This basically defines an equivalence relation, as we stated above.*

Corollary 3.8. *Any topological space $(X, \mathcal{O}_X)$ can be partitioned into connnected components. This follows from the fundamental theorem on equivalence relations, treated in the appendix. Namely, the quotient set, whose elements are equivalence classes (connected components in our case) form a partition of X.*

The above theorem lets us define connected components as equivalence classes of points.

Definition 3.9. *Let $(X, \mathcal{O}_X)$ be a topological space. We call the equivalence classes of points on X* connected components.

Proposition 3.10. *Connected components of a topological space $(X, \mathcal{O}_X)$ are closed.*

Proof. Let H be a connected component of X. Clearly, H is connected. By the theorem from the lecture that closures of connected sets are connected, we have that $\overline{H}$ is connected. Moreover, by the definition of closure

$$H \subseteq \overline{H}.$$

And since H is the maximal connected set, it follows that $H = \overline{H}$, that is, H is closed. □

Proposition 3.11. *Connected components of a topological space $(X, \mathcal{O}_X)$ are connected.*

Proof. Let y be an element of the equivalence class of x, which is denoted by $[x]$. This implies:

$$\exists A \text{ connected} : x \in A, y \in A.$$

We now show that A is actually the union of all connected subsets, which contain x. Clearly, A is a subset of the union, because it is a connected subset, which contains x. Conversely, let y be in the union of all connected subsets, which contains x. Then y is in some connected subset, which contains both x and y. Hence, y is in the equivalence class of $[x]$. We thus have:

$$[x] = \bigcup_i U_i,$$

where U_i are the connected subsets containing x. We now apply the lemma from the lecture, that union of intersecting connected sets is connected. We obtain:

$$[x] \text{ is connected.}$$

Hence, the equivalence classes, which are the connected components, are connected, as desired. □

Corollary 3.12. *The connected components are in fact the maximal connected subsets of X containing x.*

Proof. Recall that a $C \subseteq X$ is a maximal connected subset if it is connected and $if D \subset C$, then D is

not connected.

Now, proceed by contradiction. Suppose $[x]$ is not maximal. Then, there exists $D \subset X$, that is strictly larger than $[x]$, which is connected and contains x. But we showed that:

$$[x] = \bigcup_i U_i.$$

Hence, D is some U_i. But this is a contradiction, because this shows that D is not strictly larger than $[x]$. □

Remark 3.13. *Note that some textbooks define connected components of x as maximal connected subsets containing x. We have just showed that our definition is equivalent to this formulation.*

3.4 Quasi Components

This subchapter is inspired by exercise 10 from Munkres' chapter on connectedness.

Definition 3.14. *Let $(X, \mathcal{O}_X)$ be a topological space. We say x and y are quasi-equivalent,i.e. $x \sim y$ if there doesn't exist a seperation $X = A \cup B$ of X into two disjoint open sets $x \in A, y \in B$.*

Remark 3.15. *This is the same the definition we gave in the lecture. However, we want to show something stronger, namely that quasi-components are equivalence classes of points, which are quasi-equivalent.*

Proposition 3.16. *Being quasi-equivalent is an equivalence relation on any topological space $(X, \mathcal{O}_X)$.*

Proof. We check the axioms of an equivalence relation:

1. Reflexive: $\forall x \in X : x \sim x$. This holds, since there is no separation $X = A \cup B$ into disjoint open sets, such that $x \in A, x \in B$. Clearly, this can not happen, because if $x \in A, x \in B$, then A, B can't be disjoint.

2. Symmetric: $\forall x, y \in X : x \sim y \implies y \sim x$. Suppose $x \sim y$. Then there is no separation $X = A \cup B$, such that $x \in A, y \in B$, A, B are open and disjoint. However, since the union is symmetric, we have that there isn't a separation $X = B \cup A$ either. Thus, as desired, $y \sim x$.

3. Transitive: $\forall x, y, z \in X : x \sim y, y \sim z \implies x \sim z$. Suppose $x \sim y, y \sim z$. Then, there is no separation $X = A \cup B$ and there is no separation $X = C \cup D$, such that $x \in a, y \in B, y \in C, z \in D$. We prove that $x \sim z$ by contradiction. Suppose there is a separation $X = E \cup F, x \in E, z \in F$, such that E, F are open and disjoint. This means, that , since $y \in X$, in particular, that either $y \in F$ or $y \in E$. We check first the case $y \in F$. Since we know that $x \in E$, then we found a separation of y and x. This contradicts that $x \sim y$. Similarly, if $y \in E$, then we have a contradiction with $y \sim z$. Thus, as desired. $x \sim z$.

$\square$

Corollary 3.17. *We can define quasi-components, which are equivalence classes under $\sim$.*

3.5 Paths and Path connectedness

Definition 3.18. *Let $(X, \mathcal{O}_X)$ be a topological space and $x, y \in X$. The points x, y are said to be* path equivalent, *if there exists a continuous path $\gamma : [a,b] \to X : \gamma(a) = x, \ \gamma(b) = y$.*

Remark 3.19. *It is obvious, but let us mention that in the above definition continuity is meant with respect to the subspace topology on $[a,b]$ obtained from $\mathbb{R}$.*

Lemma 3.20. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and let $y \in Y$ be a point. Then the constant map:*

$$f_y : X \to Y, \ f_y(x) = y$$

is continuous.

Proof. By definition, we have that $x \in X \implies f_y(x) = y$. We thus have:

$$f_y^{-1}(y) = X$$

and for any $z \in Y : z \neq y$

$$f_y^{-1}(z) = \varnothing.$$

Now take an open set U containing y. Then:

$$f_y^{-1}(U) = X.$$

Similarly, for any open set V not containing y:

$$f_y^{-1}(V) = \varnothing.$$

Hence, the preimage of every open set is either the empty set or the whole set. Clearly, this implies that the constant map is continuous. $\square$

Lemma 3.21. *Path equivalence is an equivalence relation on any topological space $(X, \mathcal{O}_X)$.*

Proof. We have to verify that it is reflexive, symmetric, transitive:

1. reflexive: $\forall x \in X : x \sim x$. Take the constant path $\gamma : [0,1] \to X, \ \gamma(\lambda) = x \ \forall \lambda \in [0,1]$, which is continuous by which is continuous by lemma 3.20.

2. symmetric: $\forall x, y \in X : x \sim y \implies y \sim x$. Suppose $x \sim y$. Then, there exists a path

$$\gamma : [0,1] \to X : \gamma(0) = x, \gamma(1) = y.$$

Now consider the map

$$g : [0,1] \to [0,1],\ g(\lambda) := 1-\lambda,$$

which is clearly continuous. Thus, we have a new path:

$$\rho : [0,1] \to X,\ \rho(\lambda) := (\gamma \circ g)(\lambda) = \gamma(g(\lambda)) = \gamma(1-\lambda),$$

which is a path, since composition of continuous maps is continuous. Moreover, we have that:

$$\rho(0) = \gamma(1-0) = \gamma(1) = y$$

and

$$\rho(1) = \gamma(1-1) = \gamma(0) = x.$$

We thus found a path connecting y to x, namely $\rho = \gamma \circ g$. This shows that $x \sim y \implies y \sim x\ \forall x, y \in X$, as x, y were arbitrary.

3. transitive: $\forall x, y, z \in X : x \sim y$ and $y \sim z \implies x \sim z$. This is a bit trickier. Assume $x \sim y$ and $y \sim z$. We then have two paths:

$$f : [0,1] \to X : f(0) = x,\ \ f(1) = y$$

and

$$g : [0,1] \to X : g(0) = y,\ \ g(1) = z.$$

We now want to find a path from x to z. We claim the following does the job:

$$h : [0,1] \to X,\ \ h(\lambda) = \begin{cases} f(2\lambda) & \text{if } \lambda \in \left[0, \frac{1}{2}\right], \\ g(2\lambda - 1) & \text{if } \lambda \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

To see this, first let's understand why it's well-defined. On $\left[0, \frac{1}{2}\right] \cap \left[\frac{1}{2}, 1\right] = \left\{\frac{1}{2}\right\}$ we have:

$$f\left(2\frac{1}{2}\right) = f(1) = y;$$

$$g\left(2\frac{1}{2} - 1\right) = g(1-1) = g(0) = y.$$

Hence, as desired h is well-defined. Moreover, if $\lambda \in \left[0, \frac{1}{2}\right]$, then the restriction of h to this interval reads:

$$h|_{[0,\frac{1}{2}]} : \left[0, \frac{1}{2}\right] \to X,\ \ h|_{[0,\frac{1}{2}]}(\lambda) = (f \circ k)(\lambda),$$

where $k : \left[0, \frac{1}{2}\right] \to \left[0, \frac{1}{2}\right]$, $k(\lambda) = 2\lambda$, which is clearly continuous. Thus, the restriction of h to $\left[0, \frac{1}{2}\right]$ is continuous aswell, using continuity of f and that composition of continuous maps is continuous. Similarly, if $\lambda \in \left[\frac{1}{2}, 1\right]$, we have:

$$h|_{[\frac{1}{2},1]} : \left[\frac{1}{2}, 1\right] \to X,\ \ h|_{[\frac{1}{2},1]}(\lambda) = (f \circ l)(\lambda),$$

where $l : \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$, $l(\lambda) = 2\lambda - 1$, which is clearly continuous. Thus, $h|_{\left[\frac{1}{2},1\right]}$ is continuous. We finally conclude that h is continuous, thus it is a path. It is left to see that it is a path from x to z. Clearly:

$$h(0) = f(0) = x, \ \ h(1) = g(2-1) = g(1) = z.$$

Thus, as desired, we found a path connecting x to z, namely h. This shows that $x \sim y$ and $y \sim z \implies x \sim z$.

□

The above proof lets us introduce new terminology.

1. We call the path-equivalence classes of points the path- components of X.
2. We denote by $\pi_0(X)$ the set of path components of X.

Remark 3.22. *As we have seen, the elements of the set $\pi_0(X)$ are equivalence classes under the equivalence relation of being path-equivalent.*

Proposition 3.23. *Path components of a topological space $(X, \mathcal{O}_X)$ are path connected.*

Proof. Let $[x]$ be a path component and let $y, z \in [x]$. By definition, we have, that there exists a path from y to x and a path from z to x, or more formally:

$$y \sim x, \ \ z \sim x.$$

Since $\sim$ is an equivalence relation, in particular, it is symmetric:

$$z \sim x \implies x \sim z.$$

Thus, we obtain:

$$y \sim x, \ \ x \sim z.$$

By transitivity, this shows that $y \sim z$. Since y, z were arbitrary, this finishes the proof. □

Proposition 3.24. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topoloigcal spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is path-connected, then so is $(Y, \mathcal{O}_Y)$.*

Proof. We have to show that $(Y, \mathcal{O}_Y)$ is path-connected. To this end, let $y_0, y_1 \in Y$. We must find a path, which connects them. As f is a homeomorphism, it is in particular surjective. Therefore, we can find $x_0, x_1 \in X$, such that $f(x_0) = y_0, f(x_1) = y_1$. Moreover, as $(X, \mathcal{O}_X)$ is path-connected, there exists a path in X connection x_0, x_1:

$$\gamma : [0, 1] \to X : \gamma(0) = x_0, \ \ \gamma(1) = x_1.$$

As f is continuous, so will be its composition with γ. At the same time:

$$f \circ \gamma : [0,1] \to Y, \ (f \circ \gamma)(0) = f(\gamma(0)) = f(x_0) = y_0, \ (f \circ \gamma)(1) = f(\gamma(1)) = f(x_1) = y_1.$$

We thus see that $f \circ \gamma$ is a path in $(Y, \mathcal{O}_Y)$ connecting y_0 and y_1, as desired. □

Corollary 3.25. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism between them. X is path-connected iff Y is.*

Corollary 3.26. *Being path-connected is a topological property.*

Proposition 3.27. *Path-connectedness is not hereditary.*

Proof. Consider $X = \mathbb{R}$ equipped with the standard topology. Then, clearly, it is path-connected, as for any $a, b \in \mathbb{R}$, the path

$$\gamma : [0,1] \to \mathbb{R}, \ \gamma(t) := a(1-t) + bt$$

is continuous and connects a, b, since $\gamma(0) = a, \gamma(1) = b$. However, the subset $[0,1] \cup [2,3]$ equipped with the subspace topology is not path-connected. □

3.6 Functoriality of path components

Theorem 3.28. *A continuous map $f : X \to Y$ induces a well-defined set map $\tilde{f} : \pi_0(X) \to \pi_0(Y), \tilde{f}([x]_X) := [f(x)]_Y$ between the path components of X and Y. Moreover, if $g : Y \to Z$ is another continuous map, then:*

1. $\tilde{g} \circ \tilde{f} = \widetilde{g \circ f}$;
2. $\widetilde{id_X} = id_{\pi_0(X)}$.

Proof. Let $f : X \to Y$ be a continuous map and $x_1 \sim_X x_2$, that is: there exists a path $\gamma : [0,1] \to X$, such that $\gamma(0) = x_1, \gamma(1) = x_2$. We now have to show that $f(x_1) \sim_Y f(x_2)$, which means that the set map $\tilde{f}$ on equivalence classes is well-defined. Clearly, this holds, since there is such a path, namely:

$$\rho : [0,1] \to Y, \ \rho(0) = f(x_1), \ \rho(1) = f(x_2),$$

or in other words $\rho = f \circ \gamma$ is a path in Y, which connects $f(x_1)$ and $f(x_2)$. This is continuous, as γ and f are continuous, and composition of continuous maps is continuous. Moreover, if $g : Y \to Z$ is another continuous map, it induces a well-defined set map $\tilde{g} : \pi_0(Y) \to \pi_0(Z)$ in the same fashion as above. We are now ready to apply lemma D.8. Using this, we immediately get that $\tilde{g} \circ \tilde{f} = \widetilde{g \circ f}$. We also have that:

$$\widetilde{id_X} : \pi_0(X) \to \pi_0(X), \ \widetilde{id_X}([x]_X) = [id_X(x)]_X = [x]_X = id_{\pi_0(X)}([x]_X),$$

which completes the proof. □

Remark 3.29. *The above theorem can be stated in the language of category theory: consider the category* Top*, where the objects are topological spaces, the morphisms are continuous maps and the category* Set*, where the objects are sets and the morphisms are set maps. Then* π_0 *being a functor means:*

$$\pi_0 : Top \to Set$$

such that the objects are mapped as

$$X \mapsto \pi_0(X)$$

and the morphisms are mapped as:

$$f : X \to Y \mapsto \pi_0(f) : \pi_0(X) \to \pi_0(Y).$$

And moreover, this mapping satisfies:

1. $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$;
2. $\pi_0(id_{Top}) = id_{Set}$.

Remark 3.30. *The same can be said for connected components. To see this, we only have to check that a continuous map* $f : X \to Y$ *induces a well-defined set map* $\tilde{f} : X/\sim_X \to Y/\sim_Y$*, where the equivalence relation is being equivalent. To this end, suppose* $x_1 \sim x_2$*, that is:*

$$\exists A \subseteq X : x_1 \in A, x_2 \in A, \ A \text{ connected.}$$

We now have to show that $f(x_1) \sim f(x_2)$*, that is:*

$$\exists B \subseteq Y : f(x_1) \in B, f(x_2) \in B; \ B \text{ connected}$$

This clearly holds, since we can take $B = f(A)$ *and use that the image of a connected set under a continuous map is connected.*

Remark 3.31. *This is often overlooked, as it is not useful in practice, but functoriality holds for quasi components too. Suppose* x_1 *and* x_2 *are quasi-equivalent* $x_1 \sim x_2$*, that is, there does not exist a separation* $X = A \cup B$*, such that* A, B *are disjoint open sets and* $x_1 \in A, x_2 \in B$*. Now suppose* $f : X \to Y$ *is a continuous map. We have to show that there does not exist a separation* $Y = C \cup D$*, such that* C, D *are disjoint open sets in* Y *and* $f(x_1) \in C, f(x_2) \in D$*. Let's prove this by contrapositive, namely:* $f(x_1) \not\sim_Y f(x_2) \implies x_1 \not\sim x_2$*. To this end, suppose there exists such a separation, that is:*

$$\exists C, D \subseteq Y : Y = C \cup D, f(x_1) \in C, f(x_2) \in D, \ C, D \text{ open, disjoint.}$$

If this were true, this would contradict the fact that $x_1 \sim x_2$, since by continuity $f^{-1}(C), f^{-1}(D)$ are open, and preimage of disjoint sets are disjoint. Hence, we would get a separation of $X = f^{-1}(C) \cup f^{-1}(D)$, such that $x_1 \in C, x_2 \in D$, where C, D are disjoint.

Proposition 3.32. *Let $f : X \to Y$ be a continuous, surjective map. Then $\pi_0(f)$ is surjective.*

Proof. Recall that $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$. We now show that for every $[y]_Y \in \pi_0(Y)$ there exists $[x]_X \in \pi_0(X) : \pi_0(f)([x]_X) = [y]_Y$. First, by f being continuous, we know that $\pi_0(f)$ is well defined by lemma D.8. Moreover, since f is surjective, we have:

$$\forall y \in Y : \exists x \in X : f(x) = y.$$

Then, we have by the definition of $\pi_0(f)$:

$$\pi_0(f)([x]_X) = [f(x)]_Y = [y]_Y.$$

Thus, given for all $y \in Y$ there exists $x \in X : f(x) = y$, we have that for all $[y]_Y \in \pi_0(Y)$ there exists $[x]_X \in \pi_0(X)$, such that $\pi_0(f)([x]_X) = [y]_Y$, that is, $\pi_0(f)$ is surjective. $\square$

3.7 Locally path connectedness as topological property

Proposition 3.33. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is locally path-connected, then so is $(Y, \mathcal{O}_Y)$.*

Proof. As $(X, \mathcal{O}_X)$ is locally path-connected:

$$\forall x \in X : \exists U_x \in \mathcal{O}_X : x \in U_x : \forall y, z \in U_x : \exists \gamma : [0,1] \to U_x \text{ continuous} : \ \gamma(0) = y, \gamma(1) = z.$$

As f is a homeomorphism, it is in particular an open map. Hence, it maps the open set U_x, which contains x into an open set $f(U_x)$ in $\mathcal{O}_Y$, which contains $f(x)$. Moreover, f is continuous and bijective, hence if we consider $(f \circ \gamma)$, this is a path in Y which connects $f(y)$ to $f(z)$. $\square$

Corollary 3.34. *Being locally-path connected is a topological property.*

4 Problem sheet 3

4.1 Basic Operations And Connectedness

Let A, B be subsets of a topological space $(X, \mathcal{O}_X)$. Prove or disprove each of the following statements:

1. If A is connected, then so is its interior A°.
2. If A° is connected, then so is A.
3. If A is connected, then so is its closure $\bar{A}$.
4. If A is connected, then so is its boundary ∂A.
5. If ∂A is connected, then so is A.
6. If $\bar{A}$ is connected, then so is A.
7. If A, B are connected, then so is their intersection $A \cap B$.
8. If A, B are connected, then so is their union $A \cup B$.

Solution 4.1.

1. This is false. Let $X = \{a,b,c,d\}$ and $\mathcal{O}_X := \{X, \varnothing, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}\}$. Let $A := \{a,b,c\}$. Then A is connected, but the interior of A is
$A^{\mathrm{o}} = \{a,b\}$, which is disconnected, because a is clopen.
2. This is false. we consider $X = \mathbb{R}$ and $A = \mathbb{Q} \cup (-\infty, 0)$. Then $A^{\mathrm{o}} = (-\infty, 0)$ is an open interval, hence connected. But $A = (A \cap (-\infty, \pi) \cup (A \cap (\pi, \infty))$, which is disconnected.
3. This is true, the theorem was proven in the lecture.
4. This is false. Let $X = \mathbb{R}$ and $A = [0,1]$. Then A is connected, since it is a closed interval, which is connected. But the boundary $\partial A = \{0\} \cup \{1\}$, which is disconnected.
5. Let Let $X = \mathbb{R}$ and let $A = \mathbb{R} \setminus \{0\}$. Then $\partial A = \{0\}$, which is connected, as it is a singleton. But A is not connected, since $(0, \infty)$ is clopen in $\mathbb{R} \setminus \{0\}$.
6. Let $X = \mathbb{R}$ and let $A = \mathbb{R} \setminus \{0\}$. Then $\overline{A} = \mathbb{R}$, which is connected. But $(0, \infty)$ is clopen in $\mathbb{R} \setminus \{0\}$, hence A is disconnected.
7. This is false. Consider $X = \mathbb{R}^2$ and $A = \{(x,y) | x^2 + y^2 = 1\}$, $B = \{(x,y) | (x-1)^2 + y^2 = 1\}$. Since A is the unit circle and B is the circle of radius 1 around $(1,0)$, they are both connected. Their intersection is $A \cap B = \left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \right\}$ is disconnected, since the subset $\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right) \right\}$ is clopen.
8. This is false. Let $X = \mathbb{R}$ and $A = [-1,1], B = [5,6]$ respectively. Because A and B are intervals, they are connected. But $A \cup B = [-1,1] \cup [5,6]$ is disconnected.

4.2 The circle

We define the circle as the set

$$S^1 := \{z \in \mathbb{C} | |z| = 1\} \subset \mathbb{C} \cong \mathbb{R} \times \mathbb{R}$$

of unit complex numbers with subspace topology.

1. Show that S^1 fits into the pushout square

$$\begin{tikzcd} \{0,1\} \arrow[r, hook] \arrow[d] & {[0,1]} \arrow[d] \\ \{\star\} \arrow[r] & S^1 \end{tikzcd}$$

2. Show that the following spaces are not pairwise homeomorphic:

   (a) the circle S^1;

   (b) the interval $[0,1]$;

   (c) the square $[0,1] \times [0,1]$.

Solution 4.2.

1. First, recall what it means to be a pushout square:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

,where $q : A \to Y, g : Y \to Z, p : A \to X, f : X \to Z$ are continuous maps, which satisfy $f \circ p = g \circ q$.

   In our case, we have:

   (a) $A := \{0,1\}$, $Y := [0,1]$, $X := \{\star\}$, $Z := S^1$;

   (b) $q := i_1 : \{0,1\} \to [0,1] : i_1(0) := 0,\ i_1(1) := 1$; $f := i_2 : \{\star\} \to S^1, i_2(\star) := 1$;

   (c) $p : \{0,1\} \to \{\star\} : p(0) := \star, p(1) := \star$; $g := e : [0,1] \to S^1, e(t) := e^{2\pi it}$.

   We now have to check that the things we defined indeed form a pushout square. To this end, we check continuity of each map:

   (a) i_1 is clearly continuous, as $\{0,1\}$ is a discrete space.

   (b) i_2 is clearly continuous, as $\{\star\}$ is a discrete space.

   (c) p is clearly continuous, as $\{0,1\}$ is a discrete space.

   (d) e is clearly continuous, as it is the exponential function.

Moreover, we have:

$$(i_2 \circ p)(0) = i_2(p(0)) = i_2(\star) = 1, \ (i_2 \circ p)(1) = i_2(p(1)) = i_2(\star) = 1.$$

$$(g \circ i_1)(0) = g(i_1(0)) = g(0) = e^{2\pi i 0} = 1, \ (g \circ i_2)(1) = g(i_2(1)) = g(1) = e^{2\pi i 1} = 1.$$

Which shows that $i_2 \circ p = g \circ i_1$, as desisred, i.e. the diagram commutes. Finally, this means this is a pushout square. We now have to show the universal property: Given another topological space Z and continuous maps $h : \{\star\} \to Z, k : [0,1] \to Z$ such that $h \circ p = k \circ i_1$, there exists a unique $m : S^1 \to Z$ such that $m \circ i_2 = h$ and $m \circ e = k$, i.e. the pushout square commutes. Since the exponential map is surjective, m is uniquely defined by the equation $m(e^{2\pi i t}) = k(t)$. Clearly, m is well-defined, since:

$$m(e^0) = g(0) = k \circ i_1(0) = (h \circ p)(0) = h(p(0)) = h(\star) = (h \circ p)(1) = (k \circ i_1)(1) = k(1) = m(e^{2\pi i}).$$

We can also see that:

$$(m \circ i_2)(\star) = m(e^0) = m(1) = h(\star) \implies m \circ i_2 = h.$$

Moreover, by definition of m, we have:

$$(m \circ e)(t) = m(e(t)) = m(e^{2\pi i t}) = k(t) \implies m \circ e = k.$$

Hence, m is a well-defined set map, which satisfies the properties it should. We now have to show it is continuous. Since $[0,1]$ is compact and S^1 is a Hausdorff space, e is a closed map. Now, let $A \subseteq Z$ be closed. Then $k^{-1}(A)$ is closed in $[0,1]$ by continuity of k and hence $m^{-1}(A) = e(k^{-1}(A))$ is closed. Thus, m is continuous, which proves that S^1 together with e, i_2 satisfies the universal property of a pushout given the space $\{0,1\}, \{\star\}, [0,1]$ and maps i_1, p.

2. (a) We do the proof by contradiction. Suppose there is a homeomorphism $h : [0,1] \to S^1$. Then $h^{-1}\left(S^1 \setminus \left\{h\left(\frac{1}{2}\right)\right\}\right)$ is connected, as it is the image of a connected set under a continuous map. But we also know that:

$$h^{-1}\left(S^1 \setminus \left\{h\left(\frac{1}{2}\right)\right\}\right) = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right],$$

which is disconnected. Hence there can not exist a homeomorphism between $[0,1]$ and S^1.

   (b) We do the proof by contradiction. Suppose there is a homeomorphism $f : [0,1] \to [0,1] \times [0,1]$. Then $f^{-1}\left([0,1] \times [0,1] \setminus f\left(\frac{1}{2}\right)\right)$ is connected, as it is the image of a connected set under a continuous map. On the other hand:

$$f^{-1}\left([0,1] \times [0,1] \setminus f\left(\frac{1}{2}\right)\right) = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$$

is disconnected, which gives a contradiction.

(c) Suppose there is a homeomorphism $g : S^1 \to [0,1] \times [0,1]$. Then $g^{-1}([0,1] \times [0,1] \setminus g(-1,1))$ is connected, as image of a connected set under a continuous map. However:

$$g^{-1}([0,1] \times [0,1] \setminus g(-1,1)) = S^1 \cap \mathbb{R} \times (0,\infty) \cup S^1 \cap \mathbb{R} \times (-\infty,0),$$

is disconnected, which gives a contradiction.

4.3 Connected products

Show that for a family $(X_i)_{i\in I}$ of non-empty topological spaces the product $\prod\limits_{i\in I} X_i$ is connected if and only if all X_i are connected.

Solution 4.3.

"$\Leftarrow$": Suppose $\prod\limits_{i\in I} X_i$ is connected. We now need the axiom of choice to proceed, as if we would not assume it, the product could be empty, even if all of X_i are non-empty and not even connected. That would be a huge problem, as we could obtain a connected set from the product of disconnected ones. This happens if the product is empty, since the empty set is clearly connected. So let us assume that the product is non-empty, i.e. that the axiom of choice holds. Moreover, if the axiom of choice holds, we can apply theorem C.5, which implies that the projection maps $pr_i : \prod\limits_{i\in I} \to X_i$ are surjective. We also know that they are continuous by the universal property of product topology. Thus, X_i are the images of a connected set under continuous maps, hence they are connected.
"$\Rightarrow$": Suppose $X := \prod\limits_{i\in I} X_i$, where X_i are connected and non-empty for all $i \in I$. We now fix $\hat{x}_i$ for every $i \in I$. Let $J \subset I$ be a finite subset and define

$$Y_i := \begin{cases} X_i & \text{if } i \in J, \\ \{\hat{x}_i\} & \text{if } i \in I \setminus J \end{cases}.$$

Then, the cartesian product

$$\prod_{i\in I} Y_i = \prod_{i\in J} X_i \times \prod_{j\in J\setminus I} \{\hat{x}_j\}$$

is a subspace of $\prod\limits_{i\in I} X_i$, which is homeomorphic to $\prod\limits_{i\in J} X_i$. We now claim that $\prod_{i\in I} Y_i$ is connected, as it is the cartesian product of finitely many connected spaces. We already know from the lecture, theorem 3.11 that $X \times Y$ is connected if and only if X and Y are connected. We now want to show that $\prod\limits_{i=1}^{n}$ is connected iff X_i is connected for all $i = 1, \ldots, n$. We prove this by induction. The first step of the induction has been shown in the lecture. The induction hypothesis is:

$$\prod_{i=1}^{n} X_i \text{ is connected if and only if } X_i \text{ is connected for all } i = 1, \ldots, n.$$

We want to show that:

$$\prod_{i=1}^{n+1} X_i \text{ is connected if and only if } X_i \text{ is connected for all } i = 1, \ldots, n+1.$$

To see this, recall that by the definition of cartesian product, we have:

$$\prod_{i=1}^{n+1} X_i = \left(\prod_{i=1}^{n} X_i\right) \times X_{n+1}.$$

The induction hypothesis tells us that $\prod_{i=1}^{n} X_i$ is connected if and only if X_i is connected for all $i = 1, \dots, n$. On the other hand, theorem 3.11 from the lecture tells us that $\prod_{i=1}^{n+1} X_i$ is connected if and only if $\prod_{i=1}^{n} X_i$ and X_{n+1} are connected. Combining the two assertions, we obtain:

$$\prod_{i=1}^{n+1} X_i \text{ is connected if and only if } X_i \text{ is connected for all } i = 1, \dots, n+1,$$

which is exactly what we wanted to show. By proposition 3.4, we have that

$$Y := \bigcup_{J \subset I} \prod_{i \in I} Y_i$$

is connected, as it is a union of connected sets having in common a same point. We will now show that Y is dense in $X = \prod_{i \in I} X_i$, that is: $\overline{Y} = \prod_{i \in I} X_i$. Having this, we can conclude that $\prod_{i \in I} X_i$ is connected by theorem 3.8. Now to see that Y is dense in X, let $\prod_{i \in I} \{x_i\} \in \prod_{i \in I} X_i$ and consider a neighbourhood V of that point in the product topology. Then V contains an open neighbourhood U of the same point of the form $U = \prod_{i \in I} U_i$, where for all $i \in I \setminus J$, we have that $U_i = X_i$ according to remark 2.21. We then have that

$$\prod_{i \in J} \{x_i\} \times \prod_{i \in I \setminus J} \{\hat{x}_i\} \in \left(\prod_{i \in I} Y_i \cap U\right).$$

Therefore $Y \cap U \neq \varnothing$ and it follows that $\prod_{i \in I} \{x_i\} \in \overline{Y}$, which is what we wanted to show.

Remark 4.4. *We can see that for this implication the axiom of choice is not necessary. If all the X_i are non-empty and connected for $i \in I$, then the product is always connected, even if it is empty.*

4.4 (Path)-connectedness for finite spaces

Let $(X, \mathcal{O}_X)$ be a finite topological space. Show that it is path-connected iff it is connected.

Solution 4.5.

Our strategy is as follows: we will show it is locally path-connected, then that it is path-connected and use theorem 3.26 from the lecture to conclude that it is connected iff it is path-connected.
We show that $(X, \mathcal{O}_X)$ is locally-path connected. Let $x \in X$ be arbitrary. We then define:

$$B_x := \cap \{U \subseteq X | x \in U,\ U \text{ open}\}.$$

As the topological space is finite, this is a finite intersection, hence U_0 is open as a finite intersection

of open sets. Moreover, the collection $\{B_x | x \in X\}$ clearly forms a basis of X: let $x \in U, U$ open. Then, this is in the collection, as it is one of the elements of the intersection. By remark 3.22, if every element of this basis is path-connected, then X is locally path-connected. To show that for each $x \in X, B_x$ is path-connected, we have to show that there exists a path between any of its two points. To this end, let $y \in B_x$ arbitrary, then define:

$$\gamma : [0,1] \to B_x, \ \ \gamma(t) = \begin{cases} x & \text{if } t \in \left[0, \frac{1}{2}\right], \\ y & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Clearly, $\gamma(0) = x, \gamma(1) = y$. We now have to show that this is continuous. To this end, let $U \subseteq B_x$ be open. We then have 2 cases:

1. $x \in U$;

2. $X \notin U$.

Case 1 is easy, as there can't be smaller open sets in B_x containing x. It is the smallest one, as we defined it by taking the intersection of all the open sets containing x. Thus, if $x \in U$, we have $U = B_x$. This immediately tells us that $\gamma^{-1}(U) = \gamma^{-1}(B_x) = [0,1]$, which is clearly open. On the other hand, for case 2 we have that $x \notin U$. However, this still has two subcases, namely whether $y \in U$ or $y \notin U$. If $y \in U, x \notin U$, we have:

$$\gamma^{-1}(U) = \left(\frac{1}{2}, 1\right],$$

which is clearly open. Moreover, if $y \notin U, x \notin U$, then:

$$\gamma^{-1}(U) = \varnothing,$$

which is open. Thus, preimage under γ of every open set in B_x is open in $[0,1]$, which proves γ is continuous, hence the path that does the job. We conclude: B_x is path-connected. Now, by using remark 3.22 from the lecture, X is locally path-connected. Finally, using theorem 3.26, X is path-connected iff it is connected.

5 Handout4

5.1 Separating neighbourhoods

Lemma 5.1. $x \in U \iff \{x\} \subseteq U$.

Proof. "$\Rightarrow$": Suppose $x \in U$. Then, by definition of subset, we have:

$$\{x\} \subseteq U \iff \forall y \in \{x\} : y \in U.$$

By definition of $\{x\}$, its only element is x. Hence:

$$\{x\} \subseteq U \iff x \in U.$$

But we know that $x \in U$, by assumption, so we can conclude that $\{x\} \subseteq U$.
"$\Leftarrow$": Suppose that $\{x\} \subseteq U$. By definition of subset, we have:

$$\{x\} \subseteq U \iff \forall y \in \{x\} : y \in U.$$

Since x is the only element of $\{x\}$, we conclude:

$$\{x\} \subseteq U \iff x \in U.$$

$\square$

Proposition 5.2. *Let $(X, \mathcal{O}_X)$ be a topological space. Then the following hold:*

1. *$(X, \mathcal{O}_X)$ is normal $\implies$ $(X, \mathcal{O}_X)$ is regular;*
2. *$(X, \mathcal{O}_X)$ is regular $\implies$ $(X, \mathcal{O}_X)$ is Hausdorff;*
3. *$(X, \mathcal{O}_X)$ is Hausdorff $\implies$ $(X, \mathcal{O}_X)$ is T_1;*
4. *$(X, \mathcal{O}_X)$ is T_1 $\implies$ $(X, \mathcal{O}_X)$ is T_0;*

Proof. 1. Suppose $(X, \mathcal{O}_X)$ is normal. Then $(X, \mathcal{O}_X)$ is T_4 and T_1. Since $(X, \mathcal{O}_X)$ is T_4, we have:

$$\forall C, D \subseteq X \text{ closed } C \cap D = \varnothing : \exists U, V \in \mathcal{O}_X : C \subseteq U, D \subseteq V, U \cap V = \varnothing.$$

Since $(X, \mathcal{O}_X)$ is T_1, by the theorem from the lecture, this is equivalent to saying every singleton $\{x\} : x \in X$ is closed. We then take $C = \{x\}$, D closed, so that $\{x\} \cap D = \varnothing$ to obtain:

$$\exists U, V \in \mathcal{O}_X : \{x\} \subseteq U, D \subseteq V, U \cap V = \varnothing.$$

Clearly, if $\{x\} \cap D = \varnothing$, it is true that $x \notin D$. Moreover by our previous lemma $\{x\} \subseteq U \iff x \in U$.

Thus, we have:

$$\forall x \in X, D \subseteq X \text{ closed }, x \notin D : \exists U, V \in \mathcal{O}_X : x \in U, D \subseteq V, U \cap V = \varnothing,$$

which is precisely what it means to be T_3. Hence, we conclude $T_4, T_1 \implies T_3, T_1$, which is equivalent to normal $\implies$ regular.

2. Suppose $(X, \mathcal{O}_X)$ is regular. Then $(X, \mathcal{O}_X)$ is T_3 and T_1. We now want to show it is Hausdorff. Pick $x, y \in X : x \neq y$. By the T_1 property, $\{x\}$ is closed $\forall x \in X$. In particular, $\{y\}$ is closed. Since we assumed $x \neq y$, we know that $x \notin \{y\}$. But by the axiom T_3, applied to $x = x, C = \{y\}$ we have:
$$\forall x \in X, \{y\} \subseteq X, x \notin \{y\} : \exists U, V \in \mathcal{O}_X : x \in U, \{y\} \subseteq V : U \cap V = \varnothing.$$
By using our previous lemma:
$$\forall x, y \in X, x \neq y : \exists U, V \in \mathcal{O}_X : x \in U, y \in V, U \cap V = \varnothing,$$
which means precisely that $(X, \mathcal{O}_X)$ is Hausdorff.

3. Suppose $(X, \mathcal{O}_X)$ is Hausdorff. Then, for all $x, y \in X : x \neq y$, we have:
$$\exists U, V \in \mathcal{O}_X : x \in U, y \in V, U \cap V = \varnothing.$$
But since $U \cap V = \varnothing$, we can conclude that:
$$x \in U \implies x \notin V;$$
$$y \in V \implies y \notin U;$$
as if it were otherwise, the intersection would be non-empty. We thus have found:
$$\forall x, y \in X, x \neq y : \exists U, V \in \mathcal{O}_X : x \in U, y \notin U,\ x \notin V, y \in V,$$
which shows that $(X, \mathcal{O}_X)$ is T_1.

4. Suppose $(X, \mathcal{O}_X)$ is T_1. This means:
$$\forall x, y \in X : x \neq y : \exists U, V \in \mathcal{O}_X : x \in U, y \notin U,\ x \notin V, y \in V.$$
In particular, we have found:
$$\forall x, y \in X : x \neq y : \exists U \in \mathcal{O}_X : x \in U, y \notin U.$$
Hence, we can conclude that $(X, \mathcal{O}_X)$ by the fact that anything or true is true.

$\square$

Proposition 5.3. *Let $(X, \mathcal{O}_X)$ be a topological space. Then, the following are equivalent:*

1. *$(X, \mathcal{O}_X)$ is T_1;*
2. *Every singleton $\{x\} \subseteq X$ is closed;*
3. *Every subset $A \subseteq X$ is the intersection of the open sets containing A.*

Proof. We will show this by showing $1 \implies 2$ and $2 \implies 3$ and $3 \implies 1$. Let us start with the first one:

$1 \implies 2$: Suppose $(X, \mathcal{O}_X)$ is T_1. Then:

$$\forall y \in X : y \neq x : \exists U_y \in \mathcal{O}_X : y \in U_y \text{ and } x \notin U_y.$$

We now use that union of open sets is open to conclude that $\bigcup_{y \in X, y \neq x} U_y$ is open. Moreover, we claim that:

$$\bigcup_{y \in X, y \neq x} U_y = X \setminus \{x\}.$$

To see this, we show both directions:

1. $\bigcup_{y \in X, y \neq x} U_y \subseteq X \setminus \{x\}$: Let $z \in \bigcup_{y \in X, y \neq x}$. Then $z \in U_y$ for some $y \in X, y \neq x$. Since $x \notin U_y$, we must have that $z \neq x$. As $U_y \subseteq X$, we have that $z \in U_y \implies z \in X$. Hence, we have:

$$z \in U_y : y \neq x, z \neq x \implies z \in X : z \neq x \implies z \in X \setminus \{x\}.$$

2. $X \setminus \{x\} \subseteq \bigcup_{y \in X, y \neq x} U_y$: Let $z \in X \setminus \{x\}$. Then $z \in X : z \neq x$. By definition of U_z, we have that $z \in U_z$ for some $z \in X$.

Thus, we showed that $X \setminus \{x\}$ is open, which means $\{x\}$ is closed.

$2 \implies 3$. Suppose every singleton $\{x\}$ is closed and let $A \subseteq X$. Since the singletons are closed, we have that $X \setminus \{x\}$ is open for all $x \in X$. Now, using that $A \subset X$ is the intersection of all the sets of the form $X \setminus \{x\}$ for $x \notin A$, we are done, since these sets are open from the discussion above.

$3 \implies 1$: Suppose each subset $A \subseteq X$ is the intersection of the open sets containing A. Then, in particular $\{x\}$ is the intersection of the open sets containing $\{x\}$. Hence, for any $y \neq x$, there is an open set containing x, but not containing y. □

5.2 Topological properties, hereditarity and all that

The main goal of topology is to classify all topological spaces up to homeomorphism. This is a really difficult task. However, there are topological invariants, that is, some properties, that are preserved under homeomorphisms. In this subchapter, we list some of them. We have already encountered this in the previous chapter: connectedness is preserved under homeomorphisms, hence it is a topological property. The similar holds for path-connectedness, local path-connectedness. They are all topological properties.

Definition 5.4. *A topological property is a property of a topological space, that is invariant under homeomorphisms.*

Remark 5.5. *A property of spaces is a topological property if whenever a space X possesses that property, then every space homeomorphic to X possesses that property aswell.*

Definition 5.6. *A property of a topological space is called hereditary, if it is inherited by all of its subspaces.*

5.2.1 Kolmogorov spaces

Proposition 5.7. *Let $(X,\mathcal{O}_X)$ be a T_0 space and $Y \subseteq X$ equipped with the subspace topology*

$$\mathcal{O}_Y := \{Y \cap U | U \in \mathcal{O}_X\}.$$

Then $(Y,\mathcal{O}_Y)$ is T_0 aswell.

Proof. Let $y_1, y_2 \in Y : y_1 \neq y_2$. As $Y \subseteq X$, we have that $y_1, y_2 \in X$ aswell. Since $(X,\mathcal{O}_X)$ is T_0, we have:

$$\exists U \in \mathcal{O}_X : y_1 \in U, y_2 \notin U \;\; or \;\; y_1 \notin U, y_2 \in U.$$

We then have:

$$y_1 \in U \cap Y, y_2 \notin U \cap Y \;\; or \;\; y_1 \notin U \cap Y, y_2 \in U \cap Y.$$

As U was open in $(X,\mathcal{O}_X)$, by the definition of the subspace topology $\mathcal{O}_Y$, we have that $U \cap Y = Y \cap U$ is open in $(Y,\mathcal{O}_Y)$. This shows that $(Y,\mathcal{O}_Y)$ is T_0. □

Proposition 5.8. *Let $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ be topological spaces and let $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a homeomorphism. If $(X,\mathcal{O}_X)$ is T_0, then so is $(Y,\mathcal{O}_Y)$.*

Proof. As f is a homeomorphism, $f^{-1} : (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ is continuous. We do the proof by contrapositive. Suppose $(Y,\mathcal{O}_Y)$ is not Kolmogorov. That is:

$$\exists y_1, y_2 \in Y : \forall U \in \mathcal{O}_Y : (y_1 \in U \;\vee\; y_2 \notin U) \;\wedge\; (y_1 \notin U \;\vee\; y_2 \in U).$$

We now apply $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ for $A = y_1 \in U \vee y_2 \notin U, B = y_1 \notin U, C = y_2 \in U$:

$$(y_1 \in U \vee y_2 \notin U) \;\wedge\; (y_1 \notin U \vee y_2 \in U) = ((y_1 \in U \vee y_2 \notin U) \wedge (y_1 \notin U)) \vee ((y_1 \in U \vee y_2 \notin U) \wedge (y_2 \in U)).$$

Using distributivity once again:

$$\left(\underbrace{(y_1 \in U \wedge y_1 \notin U)}_{=0} \vee (y_2 \notin U \wedge y_1 \notin U)) \vee ((y_1 \in U \wedge y_2 \in U) \vee \underbrace{(y_2 \notin U \wedge y_2 \in U)}_{=0}\right).$$

Using that $0 \vee$ anything is anything:

$$(y_2 \notin U \wedge y_1 \notin U) \vee (y_1 \in U \wedge y_2 \in U).$$

Hence, the negation of being Kolmogorov reads:

$$\exists y_1, y_2 \in Y : \forall U \in \mathcal{O}_Y : y_1, y_2 \in U \vee y_1, y_2 \notin U.$$

This implies:

$$\exists y_1, y_2 \in Y : \forall U \in \mathcal{O}_Y : f^{-1}(y_1), f^{-1}(y_2) \in f^{-1}(U) \vee f^{-1}(y_1), f^{-1}(y_2) \notin f^{-1}(U)$$

But we also know that $\exists y_1, y_2 \in Y : \forall U \in \mathcal{O}_Y \iff \exists f^{-1}(y_1), f^{-1}(y_2) \in X : \forall f^{-1}(U) \in \mathcal{O}_X$ by continuity. Denote:

$$x_1 := f^{-1}(y_1) \in X, x_2 := f^{-1}(y_2) \in X, f^{-1}(U) := V \in \mathcal{O}_X.$$

Hence, using continuity, we found:

$$\exists x_1, x_2 \in X : \forall V \in \mathcal{O}_X : x_1, x_2 \in V \vee x_1, x_2 \notin V,$$

that is, $(X, \mathcal{O}_X)$ is not T_0. This finishes the proof. $\square$

Corollary 5.9. *Being Kolmogorov, or equivalently T_0 is a topological property.*

Corollary 5.10. *Given two topological spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ and a homeomorphism between them, X is T_0 iff Y is T_0.*

5.2.2 Fréchet spaces

Proposition 5.11. *Let $(X, \mathcal{O}_X)$ be a T_1 space and $Y \subset X$ equipped with the subspace topology*

$$\mathcal{O}_Y = \{U \cap Y | U \in \mathcal{O}_X\}.$$

Then $(Y, \mathcal{O}_Y)$ is T_1 aswell.

Proof. Let $y_1, y_2 \in Y : y_1 \neq y_2$. As $Y \subseteq X$, we have that $y_1, y_2 \in X$ aswell. Since $(X, \mathcal{O}_X)$ is T_1, we have:

$$\exists U \in \mathcal{O}_X : y_1 \in U, y_2 \notin U \wedge y_1 \notin U, y_2 \in U.$$

We then have:

$$y_1 \in U \cap Y, y_2 \notin U \cap Y \wedge y_1 \notin U \cap Y, y_2 \in U \cap Y.$$

As U was open in $(X, \mathcal{O}_X)$, by the definition of subspace topology $\mathcal{O}_Y$, we have that $U \cap Y = Y \cap U$ is open in $(Y, \mathcal{O}_Y)$. This shows that $(Y, \mathcal{O}_Y)$ is T_1. $\square$

Proposition 5.12. *Let $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ be topological spaces and $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a homeomorphism. If $(X,\mathcal{O}_X)$ is T_1, then so is $(Y,\mathcal{O}_Y)$.*

Proof. Let $(X,\mathcal{O}_X)$ be a T_1 space. As proven in the lecture, this is equivalent to saying that all singletons $\{x\}:x\in X$ are closed. As f is a homeomorphism, it is in particular a closed map. Thus, $f(\{x\})$ is closed in $(Y,\mathcal{O}_Y)$. Moreover, as f is a bijection, the image of $\{x\}$ under f is a singleton in Y. Hence, every singleton in $(Y,\mathcal{O}_Y)$ is closed. This is equivalent to $(Y,\mathcal{O}_Y)$ being T_1. □

Corollary 5.13. *Being Fréchet, or equivalently T_1 is a topological property.*

Corollary 5.14. *Given two topological spaces $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ and a homeomorphism between them, X is T_1 iff Y is.*

5.2.3 Hausdorff spaces

Proposition 5.15. *Let $(X,\mathcal{O}_X)$ be a Hausdorff space and let $Y\subseteq X$ be equipped with the subspace topology*

$$\mathcal{O}_Y:=\{Y\cap U|U\in\mathcal{O}_X\}.$$

Then $(Y,\mathcal{O}_Y)$ is Hausdorff.

Proof. We want to show that the tuple $(Y,\mathcal{O}_Y)$ is a Hausdorff space. To this end, pick $y_1,y_2\in Y:y_1\neq y_2$. Using that $(X,\mathcal{O}_X)$ is Hausdorff:

$$\exists U_1,U_2\subseteq X:U_1,U_2\in\mathcal{O}_X,y_1\in U_1,y_2\in U_2,U_1\cap U_2=\varnothing.$$

Since $y_1\in U_1$ and $y_1\in N$, we have that $y_1\in U_1\cap N$. Similarly, as $y_2\in U_2$ and $y_2\in N$, it is true that $y_2\in U_2\cap N$. Clearly, $U_1\cap N$ and $U_2\cap N$ are open in the subspace topology $\mathcal{O}_Y$. We still have to show they are disjoint:

$$(U_1\cap N)\cap(U_2\cap N)=(U_1\cap U_2)\cap N=\varnothing\cap N=\varnothing.$$

As $y_1,y_2\in Y$ were arbitrary, this finishes the proof. □

Proposition 5.16. *Let $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ be topological spaces and $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a homeomorphism. If $(X,\mathcal{O}_X)$ is Hausdorff, then so is $(Y,\mathcal{O}_Y)$.*

Proof. We will proceed by contrapositive. Suppose $(Y,\mathcal{O}_Y)$ is not Hausdorff. That is:

$$\exists a,b\in Y:a\neq b:\forall U_Y\in\mathcal{O}_Y:a\notin U_Y\vee\forall V_Y\in\mathcal{O}_Y:b\notin U_Y\vee U_Y\cap V_Y\neq\varnothing.$$

Using that $\neg A\vee B\iff A\implies B$ for $A=a\in U_Y,B=\forall V_Y\in\mathcal{O}_Y:b\notin U_Y\vee U_Y\cap V_Y=\varnothing$, we get:

$$\exists a,b\in Y:a\neq b:\forall U_Y\in\mathcal{O}_Y:a\in U_Y\implies(\forall V_Y\in\mathcal{O}_Y:b\notin U_Y\vee U_Y\cap V_Y\neq\varnothing).$$

Doing the same for $C = b \in U_Y$ and $D = U_Y \cap V_Y \neq \varnothing$:

$$\exists a, b \in Y : a \neq b : \forall U_Y \in \mathcal{O}_Y : a \in U_Y \implies (\forall V_Y \in \mathcal{O}_Y : b \in U_Y \implies U_Y \cap V_Y \neq \varnothing).$$

We now use that $p \implies \forall U : q(U)$, where U is some statement depending on q is the same as $\forall U : (p \implies q(U))$ for $p = a \in U_Y$, $U = V_Y, q(V_Y) = b \in U_Y \implies U_Y \cap V_Y = \varnothing$. We thus get:

$$\exists a, b \in Y : a \neq b : \forall U_Y \in \mathcal{O}_Y : \forall V_Y \in \mathcal{O}_Y : (a \in U_Y \implies b \in U_Y \implies U_Y \cap V_Y \neq \varnothing).$$

We now apply that $A \implies B \implies C \iff (A \wedge B) \implies C$ for $A = a \in U_Y$, $B = b \in U_Y$, $C = U_Y \cap V_Y \neq \varnothing$:

$$\exists a, b \in Y : a \neq b : \forall U_Y \in \mathcal{O}_Y : \forall V_Y \in \mathcal{O}_Y : (a \in U_Y \wedge b \in U_Y) \implies U_Y \cap V_Y \neq \varnothing.$$

In words, this means, that there exists at least one pair of points, for which all open sets containing them are not disjoint. We now use that $\mathcal{O}_Y$ is a topology to conclude that:

$$W_Y := U_Y \cap V_Y \in \mathcal{O}_Y, \; as \; U_Y, V_Y \in \mathcal{O}_Y.$$

Moreover, we have:

$$f^{-1}(U_Y \cap V_Y) = f^{-1}(W_Y) = f^{-1}(U_Y) \cap f^{-1}(V_Y).$$

As $U_Y, V_Y, W_Y \in \mathcal{O}_Y$ and f is a homeomorphism, in particular, it is continuous, thus:

$$f^{-1}(W_Y) \in \mathcal{O}_X, \; f^{-1}(U_Y) \in \mathcal{O}_X, \; f^{-1}(V_Y) \in \mathcal{O}_X.$$

Let us denote:

$$W_X := f^{-1}(W_Y), \; U_X := f^{-1}(U_Y), \; V_X := f^{-1}(V_Y).$$

Thus $W_X, U_X, V_X \in \mathcal{O}_X$. Moreover, as f is bijective:

$$a \in U_Y \implies f^{-1}(a) \in U_X;$$

$$b \in V_Y \implies f^{-1}(b) \in U_Y.$$

By denoting $f^{-1}(a) := x, f^{-1}(b) := y$, we get:

$$\exists x, y \in X : x \neq y : \forall U_X \in \mathcal{O}_X : \forall V_X \in \mathcal{O}_X : (x \in U_X \wedge y \in U_X) \implies U_X \cap V_X \neq \varnothing,$$

which is the negation of $(X, \mathcal{O}_X)$ being Hausdorff. This finishes the proof. $\square$

Let us show an easier, alternative proof.

Proof. Let $y_1, y_2 \in Y : y_1 \neq y_2$. As f is bijective, we have that

$$f^{-1}(y_1) \in X, f^{-1}(y_2) \in X$$

are single points aswell. Let us denote them with x_1, x_2, i.e.

$$f^{-1}(y_1) := x_1, f^{-1}(y_2) := x_2.$$

Since $(X, \mathcal{O}_X)$ is Hausdorff:

$$\exists U_X, V_X \in \mathcal{O}_X : x_1 \in U_X, x_2 \in V_X, U_X \cap V_x = \varnothing.$$

As f is a homeomorphism, in particular is an open map, thus:

$$U_X, V_X \in \mathcal{O}_X \implies f(U_X) := U_Y, f(V_X) := V_Y \in \mathcal{O}_Y.$$

Moreover, $x_1 \in U_X, x_2 \in V_X \implies f(x_1) \in U_Y, f(x_2) \in V_Y$. Now recall who x_1, x_2 were:

$$x_1 = f^{-1}(y_1) \implies f(x_1) = f(f^{-1}(y_1)) = y_1 \in U_Y;$$

$$x_2 = f^{-1}(y_2) \implies f(x_2) = f(f^{-1}(y_2)) = y_2 \in V_Y.$$

We thus have found two open sets U_Y, V_Y, which contain y_1, y_2 respectively. It is only left to show that they are disjoint. Suppose $U_Y \cap V_Y \neq \varnothing$. Then:

$$\exists z \in U_Y \cap V_Y \iff \exists z \in f(U_X) \cap f(V_X) \iff \exists z \in f(U_X \cap V_X),$$

where in the last step we used injectivity to conclude $f(U_X) \cap f(V_X) = f(U_X \cap V_X)$. But we now know that $U_X \cap V_X = \varnothing$, by $(X, \mathcal{O}_X)$ being Hausdorff. We therefore conclude:

$$U_Y \cap V_Y \neq \varnothing \iff \exists z \in \varnothing,$$

which is clearly false. Hence, its negation: $U_Y \cap V_Y = \varnothing$ is true. □

Corollary 5.17. *Being Hausdorff is a topological property.*

Corollary 5.18. *Given two topological spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ and a homeomorphism between them, X is T_2 iff Y is.*

5.2.4 T_3 spaces

Proposition 5.19. *Let $(X, \mathcal{O}_X)$ be a T_3 topological space and $Y \subseteq X$ be a subspace equipped with the subspace topology*

$$\mathcal{O}_Y := \{Y \cap U | U \in \mathcal{O}_X\}.$$

Then $(Y, \mathcal{O}_Y)$ is T_3 aswell.

Proof. Let $y \in Y$ and $F \subseteq Y : y \notin F$ be a closed set in $(Y, \mathcal{O}_Y)$. By proposition 2.2, we have that there

exists W closed in $(X, \mathcal{O}_X)$, such that:

$$F = Y \cap W.$$

As $(X, \mathcal{O}_X)$ is a T_3 space, we have:

$$\exists U, V \in \mathcal{O}_X : W \subseteq U, y \in V, U \cap V = \varnothing.$$

Since $y \in V$ and $y \in Y$, we have that $y \in Y \cap V$ and moreover since $V \in \mathcal{O}_X$, we have that $Y \cap V \in \mathcal{O}_Y$. Now since $W \subseteq U$ and $U \in \mathcal{O}_X$, we have that $Y \cap U \in \mathcal{O}_Y$. Moreover $F \subseteq Y \cap U$. We only have to show that $Y \cap U$ and $Y \cap V$ are disjoint. To see this:

$$(Y \cap U) \cap (Y \cap V) = (U \cap Y) \cap (Y \cap V) = (U \cap (Y \cap Y)) \cap V = (U \cap Y) \cap V = (Y \cap U) \cap V = Y \cap (U \cap V) = Y \cap \varnothing = \varnothing.$$

Therefore, we have found two disjoint open sets in $(Y, \mathcal{O}_Y)$, which contain y and F. More formally:

$$\exists A, B \in \mathcal{O}_Y : F \subseteq A, y \in B, A \cap B = \varnothing, \ \ namely \ A = Y \cap U, B = Y \cap V.$$

This shows that $(Y, \mathcal{O}_Y)$ is T_3. □

Proposition 5.20. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is T_3, then so is $(Y, \mathcal{O}_Y)$.*

Proof. Let $y \in Y$ and $F \subseteq Y$ closed in $(Y, \mathcal{O}_Y) : y \notin F$. Clearly, $f^{-1}(F), f^{-1}(y)$ are disjoint, i.e. $f^{-1}(F) \cap f^{-1}(y) = f^{-1}(F \cap y) = f^{-1}(\varnothing) = \varnothing$, where we used that f is injective as it is a homeomorphism. As f is bijective, the preimage of y is a point in X. Since f is continuous, the preimage of F is closed in $\mathcal{O}_X$. Now, since $(X, \mathcal{O}_X)$ is T_3, we obtain:

$$\exists U, V \in \mathcal{O}_X : f^{-1}(y) \in U, f^{-1}(F) \subseteq V.$$

Now using that f is open, we have:

$$f(U), f(V) \in \mathcal{O}_Y, U \cap V = \varnothing.$$

We therefore found two disjoint open sets, which contain y and F respectively. As y, F were arbitrary, this concludes the proof. □

Corollary 5.21. *Being T_3 is a topological property.*

Corollary 5.22. *Given two topological spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ and a homeomorphism between them, X is T_3 iff Y is.*

5.2.5 T_4 spaces

Proposition 5.23. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homeomorphism. If $(X, \mathcal{O}_X)$ is T_4, then so is $(Y, \mathcal{O}_Y)$.*

Proof. Let Y_1, Y_2 be disjoint closed sets in $(Y, \mathcal{O}_Y)$. We dnote:

$$X_1 := f^{-1}(Y_1), \; X_2 := f^{-1}(Y_2).$$

This immediately implies:

$$X_1 \cap X_2 = f^{-1}(Y_1) \cap f^{-1}(Y_2) = f^{-1}(Y_1 \cap Y_) = f^{-1}(\varnothing) = \varnothing.$$

As f is a homeomorphism, it is continuous, which implies $X_1, X_2 \in \mathcal{O}_X$. As $(X, \mathcal{O}_X)$ is T_4:

$$\exists U_1, U_2 \in \mathcal{O}_X : X_1 \subseteq U_1, X_2 \subseteq U_2, U_1 \cap U_2 = \varnothing.$$

We also know that:

$$X_1 \subseteq U_1 \implies f(X_1) \subseteq f(U_1) \iff Y_1 \subseteq f(U_1);$$

$$X_2 \subseteq U_2 \implies f(X_2) \subseteq f(U_2) \iff Y_2 \subseteq f(U_2).$$

As $U_1, U_2 \in \mathcal{O}_X$ and f is open, $f(U_1), f(U_2) \in \mathcal{O}_Y$. We thus have found two open sets in $\mathcal{O}_Y$, which contain Y_1, Y_2 respectively. It is left to show that they are disjoint. This can easily be seen:

$$f(U_1) \cap f(U_2) = f(U_1 \cap U_2) = f(\varnothing) = \varnothing.$$

□

Corollary 5.24. *Being T_4 is a topological property.*

5.3 Urysohn metrization theorem

To do: prove this theorem. It is in particular important for physics, since this implies that every topological manifold is metrizable. This is very useful, as metrizable spaces satisfy all separation properties.

6 Problem sheet 4

6.1 Hausdorff spaces

1. A topological space $(X, \mathcal{O}_X)$ is Hausdorff if and only if the diagonal

$$\Delta := \{(x,x) | x \in X\}$$

is closed in the product space $(X \times X, \mathcal{O}_{X \times X})$.

2. A subspace of a Hausdorff space is a Hausdorff space.

3. If $(X, \mathcal{O}_X)$ is regular and $A \subseteq X$ closed in the subspace topology, then the quotient space X/A is Hausdorff.

Solution 6.1.

1. "$\Rightarrow$": We want to show that Δ is closed in $\mathcal{O}_{X \times X}$. This is equivalent to showing that $(X \times X) \setminus \Delta$ is open in $\mathcal{O}_{X \times X}$. By the lecture this is equivalent to the fact that $(X \times X) \setminus \Delta$ is a neighbourhood of all its points, i.e.:

$$\forall (x_1, x_2) \in (X \times X) \setminus \Delta : \exists V \subseteq ((X \times X) \setminus \Delta) : V \in \mathcal{O}_{X \times X}, (x_1, x_2) \in V.$$

To show this, let $(x_1, x_2) \in (X \times X) \setminus \Delta$. We then know that $x_1, x_2 \in X : x_1 \neq x_2$. By Hausdorffness, we thus find:

$$U_1, U_2 \in \mathcal{O}_X : x_1 \in U_1, x_2 \in U_2 : U_1 \cap U_2 = \varnothing.$$

Hence, our candidate is $V = U_1 \times U_2$, which is open in $\mathcal{O}_{X \times X}$ by the definition of product topology. It is left to show that $(U_1 \times U_2) \subseteq ((X \times X) \setminus \Delta)$. We will prove this by contrapositive. Suppose $(x_1, x_2) \notin ((X \times X) \setminus \Delta)$. Then, we have that:

$$(x_1, x_2) \in \Delta \iff x_1, x_2 \in X : x_1 = x_2.$$

If this specific (x_1, x_2) with $x_1 = x_2$ were in $U_1 \times U_2$, we would have that $x_1 \in U_1$ and $x_1 \in U_2$, which would contradict that $U_1 \cap U_2 = \varnothing$. Hence, as desired $(x_1, x_2) \notin U_1 \times U_2$.
"$\Leftarrow$: Suppose the diagonal Δ is closed in $(X \times X, \mathcal{O}_{X \times X})$. Then, its complement, $(X \times X) \setminus \Delta$ is open. We now want to show that X is Hausdorff. To this end, let $x_1, x_2 \in X : x_1 \neq x_2$. This implies $(x_1, x_2) \in (X \times X) \setminus \Delta$. As $(X \times X) \setminus \Delta$ is open, it is the neighbourhood of all its points:

$$\exists B \subseteq ((X \times X) \setminus \Delta) : B \in \mathcal{O}_{X \times X}, (x_1, x_2) \in B.$$

As B is open in the product topology, there exists $U_1, U_2 \in \mathcal{O}_X : x_1 \in U_1, x_2 \in U_2$ and $U_1 \times U_2 \subseteq B$. Therefore $U_1 \times U_2 \subseteq ((X \times X) \setminus \Delta)$, since $B \subseteq ((X \times X) \setminus \Delta)$. This implies:

$$(U_1 \times U_2) \cap \Delta = \varnothing.$$

We now want to show that this implies $U_1 \cap U_2 = \varnothing$. We proceed by contrapositive. Suppose $U_1 \cap U_2 \neq \varnothing$. We then have to show that $(U_1 \times U_2) \cap \Delta \neq \varnothing$. By the assumption that $U_1 \cap U_2 \neq \varnothing$, we have:

$$\exists y_1, y_2 \in X : (y_1, y_2) \in U_1 \times U_2 : y_1 = y_2, \ \ i.e. \ \exists y_1 \in X : (y_1, y_1) \in U_1 \times U_2.$$

Using the definition of Δ, we can conclude, that the intersection $(U_1 \times U_2) \cap \Delta$ is non-empty, as it contains at least (y_1, y_1). This finishes the proof.

2. Suppose $(X, \mathcal{O}_X)$. Then a subspace $Y \subset X$ is equipped with the subspace topology:

$$\mathcal{O}_Y = \{Y \cap U | U \in \mathcal{O}_X\}.$$

We now want to show that the tuple $(Y, \mathcal{O}_Y)$ is a Hausdorff space. To this end, pick $y_1, y_2 \in Y : y_1 \neq y_2$. Using that $(X, \mathcal{O}_X)$ is Hausdorff:

$$\exists U_1, U_2 \subseteq X : U_1, U_2 \in \mathcal{O}_X, y_1 \in U_1, y_2 \in U_2, U_1 \cap U_2 = \varnothing.$$

Since $y_1 \in U_1$ and $y_1 \in N$, we have that $y_1 \in U_1 \cap N$. Similarly, as $y_2 \in U_2$ and $y_2 \in N$, it is true that $y_2 \in U_2 \cap N$. Clearly, $U_1 \cap N$ and $U_2 \cap N$ are open in the subspace topology $\mathcal{O}_Y$. We still have to show they are disjoint:

$$(U_1 \cap N) \cap (U_2 \cap N) = (U_1 \cap U_2) \cap N = \varnothing \cap N = \varnothing.$$

As $y_1, y_2 \in Y$ were arbitrary, this finishes the proof.

3. Let $(X, \mathcal{O}_X)$ be a regular space and $A \subseteq X$ be a closed subset of X. This implies that $(X, \mathcal{O}_X)$ is T_3 and T_1 and $X \setminus A \in \mathcal{O}_X$. From now on we will denote the quotient space X/A as $X/\sim$ under the equivalence relation given in the next step. We now claim that $x \sim y :\iff x = y$ or $x, y \in A$ is an equivalence relation. To see this, we check the three axioms:

   (a) Reflexive: $\forall x \in X : x \sim x$, as $x = x$.

   (b) Symmetric: $\forall x, y \in X : x \sim y \implies y \sim x$. Suppose $x \sim y$. Then $x = y$ or $x, y \in A$. Clearly, this implies that $y = x$ or $y, x \in A$. Thus, $y \sim x$, as desired.

   (c) Transitive: $\forall x, y, z \in X : x \sim y$ and $y \sim z \implies x \sim y$. Suppose $x \sim y$ and $y \sim z$. This implies $x = y$ or $x, y \in A$ and $y = z$ or $y, z \in A$. Now suppose $x = y$. In this case $x = z$ or $x, z \in A$. On the other hand, if $x \neq y$, we have that $x, y \in A$ and $y = z$ or $y, z \in A$. This splits into further two subcases. Suppose $y = z$. In this case, $x, z \in A$. If $y \neq z$, then $x, y \in A$ and $y, z \in A$, so in particular $x, z \in A$. We can conclude: $x \sim z$.

We then have the quotient space $X/\sim$ given as:

$$X/\sim = \{[x] : x \in X\}.$$

Moreover, an equivalence class can be evaluated to give:

$$[x] = \begin{cases} A & \text{if } x \in A, \\ \{x\} & \text{if } x \notin A \end{cases}.$$

We now define the map

$$p : X \to X/\sim, x \mapsto [x] = \begin{cases} A & \text{if } x \in A, \\ \{x\} & \text{if } x \notin A \end{cases},$$

which is guaranteed to be a quotient map, given the quotient space is equipped the quotient topology. We now in principle have the following possibilities:

1. $x, y \in A : x \neq y \implies p(x) = p(y) = [x] = [y] = A$;
2. $x \in A,\ y \in X \setminus A \implies p(x) = A, p(y) = \{y\}$;
3. $x, y \in X \setminus A : x \neq y \implies p(x) = \{x\},\ p(y) = \{y\}$.

We can see that if both $x, y \in A$, then in the quotient, they are identified, that is $p(x) = p(y) = A$. Hence, if they are both in A, then they are not disjoint in the quotient. Therefore, it is enough to verify the other two cases for Hausdorffness.

1. Suppose $x \in A, y \in X \setminus A \implies p(x) \cap p(y) = A \cap \{y\} = \varnothing$. That is, the two points are disjoint in the quotient. We have to show that there exits two disjoint open sets in the quotient, which contain these points. To this end, we use that a set in the quotient topology is open iff its preimage under the projection is open in X. Computing the preimages:

   $$p^{-1}(A) = A;$$

   $$p^{-1}(\{y\}) = y.$$

   As X is regular, it is in particular T_3 and thus, since A is closed and $y \in X, y \notin A$, we can find two disjoint open sets separating them:

   $$\exists U, V \in \mathcal{O}_X : U \cap V = \varnothing, y \in U, A \subseteq V.$$

   We now claim that $p^{-1}(p(U)) = U$. To see this, we show both inclusions. Clearly $U \subseteq p^{-1}(p(U))$ as:

   $$z \in U \implies p(z) \in p(U).$$

   On the other hand, suppose $z \in p^{-1}(p(U))$. We then have:

   $$p(z) \in p(U) \implies p(z) = p(u) : u \in U.$$

   But by the definition of the map p:

   $$p(z) = p(u) \iff z \sim u \iff z = u \text{ or } z, u \in A.$$

If $z = u$, then clearly $z \in U$. The case $z, u \in A$ can not occur, as A and U are disjoint. Hence, $z \in U$. We conclude:

$$p^{-1}(p(U)) = U.$$

Moreover, as U is open in X, this implies that $p(U)$ is open in $X/\sim$, by the definition of quotient topology. Similarly, we have that $p^{-1}(p(V)) = V$. To see this, we show both inclusions. Clearly, $V \subseteq p^{-1}(p(V))$ as:

$$m \in V \implies p(z) \in p(V).$$

On the other hand, suppose $m \in p^{-1}(p(V))$. We then have:

$$p(m) \in p(V) \implies p(m) = p(v) : v \in V.$$

Using the definition of the map p:

$$p(m) = p(v) \iff p \sim v \iff m = v \;\; or \;\; m, v \in A.$$

In case $m = v$, then we clearly have that $m \in V$. In the other case, if $m, v \in A$, then $m \in V$ since $A \subseteq V$. This concludes:

$$p^{-1}(p(V)) = V.$$

Moreover, as V is open in X, this implies that $p(V)$ is open in $X/\sim$ by the definition of quotient topology.

We know that $U \cap V = \varnothing$. We then claim that $p(U) \cap p(V) = \varnothing$. To see this, we proceed by contrapositive. Suppose $p(U) \cap p(V) \neq \varnothing$:

$$\exists a \in p(U) \cap p(V) \implies \exists a \in X/\sim : a \in p(U) \wedge a \in p(V) \iff \exists a \in X/\sim : \exists u \in U : a \sim u \wedge \exists v \in V : a \sim v.$$

By reflexivity, if $a \sim u$, then $u \sim a$. By transitivity, $u \sim a$ and $a \sim v$, then $u \sim v$. We then have:

$$u \sim v \iff u = v \;\; or \;\; u, v \in A.$$

If $u = v$, then $U \cap V \neq \varnothing$. If $u, v \in A$, then in particular $u \in A$, which implies that $U \cap A \neq \varnothing$. But as $A \subseteq V$, this tells us that $U \cap V \neq \varnothing$. Finally, since $y \in U, A \subseteq V$, we have that:

$$p(y) \in p(U) \implies \{y\} \in p(U);$$

$$p(A) \subseteq p(V) \implies A \in p(V).$$

Therefore, we found two disjoint open sets containing $\{y\}$ and A, namely: $p(U), p(V)$. Formally, this can be written down as:

$$\forall \{y\} \in X/\sim : \exists p(U), p(V) \in \mathcal{O}_{X/\sim} : \{y\} \in p(U), A \in p(V) : p(U) \cap p(V) = \varnothing.$$

Remark 6.2. *As we have seen, in this case, we did not use the T_1 condition of regularity, it was not needed in the proof.*

2. Suppose $x, y \in X \setminus A : x \neq y$. We then have $\{x\}, \{y\} \in X/\sim$: $\{x\} \neq \{y\}$. We have to show that there exist two disjoint open sets in the quotient, which contain $\{x\}$ and $\{y\}$. We proceed similarly. The preimages are:

$$p^{-1}(\{y\}) = y;$$

$$p^{-1}(\{x\}) = x.$$

As X is regular, it is also Hausdorff, as was shown in proposition 5.2.

Remark 6.3. *We see that in this direction regularity is strictly needed in order to proceed with the proof.*

As X is Hausdorff, we can separate x, y:

$$\exists U, V \in \mathcal{O}_X : x \in U, y \in V : U \cap V \neq \varnothing.$$

We know that $x \in X \setminus A$ and $y \in X \setminus A$, by assumption. This implies:

$$x \in (U \cap (X \setminus A)), \;\; y \in (V \cap (X \setminus A)).$$

Remark 6.4. *We now use that A is closed, so this is also needed for this direction.*

Let us define $U_1 := U \cap (X \setminus A)$, $V_1 := V \cap (X \setminus A)$. Since A is closed, $X \setminus A$ is open, and therefore using that intersection of open sets is open, we obtain that $U_1, V_1 \in \mathcal{O}_X$. Moreover, $U_1 \cap V_1 = \varnothing$, as the intersection is associative and commutative:

$$U_1 \cap V_1 = (U \cap (X \setminus A)) \cap (V \cap (X \setminus A)) = (U \cap V) \cap ((X \setminus A) \cap (X \setminus A)) = \varnothing \cap (X \setminus A) = \varnothing.$$

We now claim that $p(U_1)$ and $p(V_1)$ are two open,disjoint sets in $X/\sim$, containing $\{x\}$ and $\{y\}$ respectively. Let us first show that they are open. Start with $p(U_1)$. We will use the same trick as before, namely: $p^{-1}(p(U_1)) = U_1$. We prove this by double inclusion, but omit the first inclusion, as it is the same as above and hence trivial. Now suppose $z \in p^{-1}(p(U_1))$. We then have:

$$p(z) \in p(U_1) \iff p(z) = p(u_1) : u_1 \in U_1.$$

By the definition of p:

$$p(z) = p(u_1) \iff z = u_1 \;\; or \;\; z, u_1 \in A.$$

In the case $z = u_1$ we immediately obtain that $z \in U_1$. The other case can be ruled out, as it says that $u_1 \in A$, which contradicts that $u_1 \in U_1$, as $U_1 \cap A = \varnothing$. This finally shows that $p^{-1}(p(U_1)) = U_1$. Moreover, as U_1 is open in X, we have that $p(U_1)$ is open in $X/\sim$ by the

definition of quotient topology. We now show that $p^{-1}(p(V_1)) = V_1$. Suppose $n \in p^{-1}(p(V_1))$:

$$p(n) \in p(V_1) \iff p(n) = p(v_1) : v_1 \in V_1;$$

By the definition of p:

$$p(n) = p(v_1) \iff n = v_1 \;\; or \;\; n, v_1 \in A.$$

In the case $n = v_1$, clearly $n \in V_1$. The other case can be ruled out, as it would contradict that $v_1 \in V_1$. Similarly as above, $p(V_1)$ is open in $X/\sim$ by the definition of quotient topology and V_1 being open in X. Moreover, we have:

$$x \in U_1 \implies p(x) \in p(U_1) \iff \{x\} \in p(U_1);$$

$$y \in V_1 \implies p(y) \in p(V_1) \iff \{y\} \in p(V_1).$$

We now have to show that $p(U_1)$ and $p(V_1)$ are disjoint to finish the proof. We proceed by contrapositive. Suppose $p(U_1) \cap p(V_1) \neq \varnothing$:

$$\exists a \in p(U_1) \cap p(V_1) \iff \exists a \in X/\sim : a \in p(U_1) \land a \in p(V_1) \iff \exists a \in X/\sim : \exists u_1 \in U_1 : a \sim u_1 \land \exists v_1 \in V_1 : a \sim v$$

By reflexivity, if $a \sim u_1$, then $u_1 \sim a$. By transitivity, $u_1 \sim a$ and $a \sim v_1$, then $u_1 \sim v_1$. We then have:

$$u_1 \sim v_1 \iff u_1 = v_1, \;\; or \;\; u_1, v_1 \in A.$$

If $u_1 = v_1$, this implies $U_1 \cap V_1 \neq \varnothing$. The other case can be ruled out, as it would contradict that $u_1, v_1 \in U_1, V_1$, as U_1 and V_1 are disjoint from A.
We thus have found two disjoint, open sets in $X/\sim$, which contain $\{x\}, \{y\}$ respectively, namely: $p(U_1), p(V_1)$. This shows that $X/\sim$ is Hausdorff.

6.2 Separation properties of products and quotients

1. For $t \in \{0,1,2,3\}$, the product $X = \prod_{i \in I} X_i$ of non-empty spaces is a T_t space if and only if all X_i are T_t spaces.

2. The quotient $\mathbb{R}/\sim$ by the equivalence relation $x \sim y$ if $x = y$ or $x, y \in \mathbb{Q}$ is neither T_1, T_2, T_3 nor T_4. Is it T_0?

Solution 6.5.

1. Before we proceed to the exercise, let us prove a lemma, which we will use.

Lemma 6.6. *Assume the axiom of choice holds. Let $(X_i, \mathcal{O}_i)_{i \in I}$ be a family of topological spaces labelled by an index set I. Let $X := \prod_{i \in I} X_i$ be the product space equipped with the product topology $\mathcal{O}$. Let $z \in X$. Then for each $i \in I$:*

$$Y_i := \{x \in X | \forall j \in I \setminus \{i\} : pr_j(x) = pr_j(z)\}$$

equipped with the subspace topology $\mathcal{O}_{Y_i}$ is homeomorphic to $(X_i, \mathcal{O}_i)$, where the homeomorphism is given by:

$$p_i : (Y_i, \mathcal{O}_{Y_i}) \to (X_i, \mathcal{O}_i), \;\; p_i(y) = y_i.$$

Proof. We will show that p_i are homeomorphisms by showing they are:

(a) bijective;

(b) continuous;

(c) open.

We show these one by one:

(a) As the axiom of choice holds, the projections pr_i are surjective. As p_i are restriction of the projections, they are surjective too. We now want to show that they are injective. To this end, let $i \in I, z \in X$. We want to show that $\forall x, y \in Y_i : p_i(x) = p_i(y) \implies x = y$. We know that:

$$x = y \iff x_j = y_j \forall j \in I.$$

As $p_i(x) = p_i(y)$, we immediately have that $x_i = y_i$ for the fixed $i \in I$, that we chose. Moreover, as $x, y \in Y_i$, we have:

$$x_j = z_j, \;\; y_j = z_j \; \forall j \in I \setminus \{i\} \implies x_j = y_j \; \forall j \in I \setminus$$

Hence, as desired we have that $x_j = y_j \forall j \in I$. This shows that $x = y$, proving injectivity. As p_i are injective and surjective, they are bijective.

(b) Let $V \in \mathcal{O}_i$ and

$$U := \prod_{i \in I} U_i, \;\; U_j = \begin{cases} X_j & \text{if } i \neq j, \\ V & \text{if } i = j \end{cases}.$$

Clearly U is open in the product space $(X, \mathcal{O})$, as it is a basis element. Now, let $x \in Y_i$. We want to show that:

$$p_i^{-1}(V) = U \cap Y_i.$$

To this end, we do a double inclusion. Let $x \in p_i^{-1}(V)$. We then have:

$$p_i(x) \in V \implies x \in U.$$

As $x \in U$ and $x \in Y_i$, we have that $x \in U \cap Y_i$. Conversely, suppose $x \in U \cap Y_i$. We then have:

$$x \in U \;\; and \;\; x \in Y_i.$$

In particular, $x \in U$, hence $p(x) \in V \iff x \in p^{-1}(V)$. We thus showed that:

$$p_i^{-1}(V) = U \cap Y_i.$$

On the other hand, we know that U is open in the product space $(X,\mathcal{O})$ and $Y_i \subseteq X$. Therefore, by the definition of subspace topology:

$$p^{-1}(V) \text{ open in } Y_i.$$

This concludes that p_i are continuous.

(c) We want to show that p_i are open maps. To this end, let A open in Y_i. We want to show that $p_i(A)$ is open in X_i. To this end, let $x \in p_i(A)$. Since p_i are surjective:

$$\exists y \in A : p_i(y) = x.$$

Moreover, by the definition of subspace topology, we have that:

$$\exists U' \in \mathcal{O} : A = U' \cap Y_i.$$

For all $k \in I$, let pr_k denote the projection to the $k-th$ factor: $pr_k : X \to X_k$. By the definition of product topology, we have that:

$$\bigcap_{k \in J} pr_k^{-1}(V_k) \subseteq U',$$

where J is a finite subset of I and $V_k \in \mathcal{O}_k$. Moreover, we have:

$$\bigcap_{k \in J} pr_k^{-1}(V_k) \subseteq U' \implies \bigcap_{k \in J} pr_k^{-1}(V_k) \cap Y_i \subseteq U' \cap Y_i = A.$$

As we know that $y \in A$, we also have that $y \in \bigcap\limits_{k \in J} pr_k^{-1}(V_k) \cap Y_i$. Now using that $x \in p_i(A)$, we have:

$$x \in p_i\left(\bigcap_{k \in J} pr_k^{-1}(V_k) \cap Y_i\right) \subseteq p_i(A).$$

Using that intersections are associative and that p_i are injective:

$$p_i\left(\bigcap_{k \in J} pr_k^{-1}(V_k) \cap Y_i\right) = p_i\left(\bigcap_{k \in J} (pr_k^{-1}(V_k) \cap Y_i)\right) = \bigcap_{k \in J} p_i\left(pr_k^{-1}(V_k) \cap Y_i\right).$$

As $p_i\left(pr_k^{-1}(V_k) \cap Y_i\right) \in \mathcal{O}_i$ and J is finite, we thus have found an open set, which contains x and is a subset of $p_i(A)$. This shows that $p_i(A)$ is a neighbourhood of x. As x was arbitrary, this shows that $p_i(A)$ is a neighbourhood of all its points, hence open. This shows that p_i are open maps.

Since p_i are open, continuous, bijective, they are homeomorphisms, as desired. □

Using this lemma, we will do each proof one by one:

(a) T_0: "$\Rightarrow$:" Suppose X is T_0. By proposition 5.7, we have that the subspace

$$Y_i := \{x \in X | \forall j \in I \setminus \{i\} : pr_j(x) = pr_j(z)\}$$

equipped with subspace topology is T_0. By the previous lemma 6.6, we have that $(Y_i, \mathcal{O}_{Y_i})$ is homeomorphic to X_i. Using proposition 5.8, which says that homeomorphisms preserve T_0 property, we see that X_i are also T_0.

"$\Leftarrow$": Suppose X_i is T_0 for all $i \in I$. Let $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I}$ be points in X, such that $x \neq y$. Then, there exists $j \in I : x_j \neq y_j$. As X_j is T_0 for all $j \in I$, there exists an open subset $U \subseteq X_j$, such that $x_j \in U$ and $y_j \notin U$ or $x_j \notin U$ and $y_j \in U$. As pr_j are continuous, $pr_j^{-1}(U)$ are open in $(X, \mathcal{O}_X)$. Moreover, it is true that:

$$\left(x \in pr_j^{-1}(U) \wedge y \notin pr_j^{-1}(U)\right) \vee \left(x \notin pr_j^{-1}(U) \wedge y \in pr_j^{-1}(U)\right).$$

(b) T_1: "$\Rightarrow$": Suppose X is T_1. By proposition 5.11, we have that the subspace

$$Y_i := \{x \in X | \forall j \in I \setminus \{i\} : pr_j(x) = pr_j(z)\}$$

equipped with the subspace topology is T_1. By lemma 6.6, it is homeomorphic to X_i. Using proposition 5.12, which says that homeomorphisms preserve T_1, we conclude that X_i are also T_1 for all $i \in I$.

"$\Leftarrow$": Suppose X_i is T_1 for all $i \in I$. Let $x = (x_i)_{i \in I}, y = (y_i)_{i \in I}$ be two distinct points in X. Then, there is $j \in I$, such that $x_j \neq y_j$. Because X_j is T_1, there are open sets $U, V \subseteq X_j$, such that $x_j \in U$ and $y_j \in U$ and $x_j \notin V, y_j \in V$. Using continuity of the projections, we find $pr_j^{-1}(U), pr_j^{-1}(V)$ are open in $(X, \mathcal{O}_X)$ and moreover:

$$\left(x \in pr_j^{-1}(U) \wedge y \notin pr_j^{-1}(U)\right) \wedge \left(x \notin pr_j^{-1}(V) \wedge y \in pr_j^{-1}(V)\right).$$

(c) T_2: "$\Rightarrow$": Suppose X is T_2. By proposition 5.15, we have that the subspace:

$$Y_i := \{x \in X | \forall j \in I \setminus \{i\} : pr_j(x) = pr_j(z)\}$$

equipped with the subspace topology is Hausdorff. By lemma 6.6, we have that $(Y_i, \mathcal{O}_{Y_i})$ is homeomorphic to X_i. Using proposition 5.16, which tells us that Hausdorffness is preserved under homeomorphisms, we conclude that X_i are Hausdorff for all $i \in I$.

"$\Leftarrow$: Suppose X_i is Hausdorff for all $i \in I$. Let $x = (x_i)_{i \in I}, y = (y_i)_{i \in I}$ be two distinct points in X. Then, there exists $j \in I$, such that $x_j \neq y_j$. Because X_j is T_2, there are open sets $U, V \subseteq X$, such that $x_j \in U, y_j \in V, U \cap V = \varnothing$. Using continuity of the canonical projections, we find $pr^{-1}(U), pr^{-1}(V)$ are open in $(X, \mathcal{O}_X)$, moreover:

$$x \in pr_j^{-1}(U), y \in pr_j^{-1}(V), pr_j^{-1}(U) \cap pr_j^{-1}(V) = \varnothing,$$

which shows that X is Hausdorff.

(d) T_3: "$\Rightarrow$": Suppose X is T_3. By proposition 5.19, we have that the subspace:

$$Y_i := \{x \in X | \forall j \in I \setminus \{i\} : pr_j(x) = pr_j(z)\}$$

equipped with the subspace topology is T_3. By lemma 6.6, we have that $(Y_i, \mathcal{O}_Y)$ is homeomorphic to X_i. Using proposition 5.20, which tells us that T_3 is preserved under homeomorphisms, we conclude that X_i are T_3 for all $i \in I$.

"$\Leftarrow$": Suppose X_i is T_3 for all $i \in I$. Let $(x_i)_{i\in I} \in X$. X being T_3 is equivalent to the fact that the closed neighbourhoods of $x \in X$ form a neighbourhood basis of x. Let $U := \prod\limits_{j\in I} U_j$ with $U_j \subseteq X_j$ open and

$J := \{j \in I | U_j \neq X_j\}$ finite be a member of the basis containing x.

Then, by X_j being T_3, we have that for all $j \in J$ there are $V_j \subseteq X_j$ open and $C_j \subseteq X_j$ closed, such that $x_j \in V_j \subseteq C_j \subseteq U_j$, because the closed sets form a neighbourhood basis of x_j in X_j. The set $W := \prod_{i\in I} W_i$ with $V_i = W_i$ for $i \in J$ and $W_i = X_i$ for $i \notin J$ is open in X. We also have that $C := \bigcup\limits_{j\in J} pr_j^{-1}(C_j)$ is closed in X, as the intersection of finitely many closed sets is closed and pr_j is continuous.

Now, $x \in W \subseteq C \subseteq U$, which shows that C is a closed neighbourhood of x contained in U. So the closed neighbourhoods of x form a neighbourhood basis in X and hence X is T_3.

2. First, we check that the relation given is indeed an equivalence relation:

(a) Reflexive: clear as $x = x$ for all $x \in \mathbb{R}$;

(b) Symmetric: suppose $x = y$. Then clearly $y = x$. Otherwise, if $x, y \in \mathbb{Q}$, then $y, x \in \mathbb{Q}$ aswell;

(c) Transitive: suppose $x = y$ and $y = z$. Then clearly $x = y = z$. Otherwise, if $x, y \in \mathbb{Q}$ and $y, z \in \mathbb{Q}$, then clearly $x, z \in \mathbb{Q}$.

We then have the quotient space given by:

$$\mathbb{R}/\sim = \{[x] | x \in \mathbb{R}\}.$$

Denote the quotient projection as:

$$q : \mathbb{R} \to \mathbb{R}/\sim, q(x) = [x].$$

(a) T_0: Let $x, y \in \mathbb{R}$, such that $x \notin \mathbb{Q}$ and $x \neq y$. This immediately implies that $q(x) \neq q(y)$, as $q(x) = q(y) \iff [x] = [y] \iff x = y \vee x, y \in \mathbb{Q}$. None of the conditions are satisfied, so $q(x) \neq q(y)$. We now want to show that there exists an open set in $\mathbb{R}/\sim$, which contains $q(y)$, but not $q(x)$ or which contains $q(x)$, but not $q(y)$. We now claim that the set

$$q^{-1}(q(\mathbb{R} \setminus \{x\})) = \mathbb{R} \setminus \{x\}$$

will do the job. First, let us show this equality by double inclusion:

i. Suppose $a \in q^{-1}(q(\mathbb{R}\setminus\{x\}))$. We then have:

$$\exists z \in q(\mathbb{R}\setminus\{x\}) : q(a) = z.$$

That is:

$$\exists z \in q(\mathbb{R}\setminus\{x\}) : [a] = z.$$

Now, suppose for the sake of contradiction that $z = [x]$. We then have that $[a] = [x] \iff a = x \vee a, x \in \mathbb{Q}$. As $x \notin \mathbb{Q}$, we must have that $a = x$. However, if $a = x$, we have that $[x] \in q(\mathbb{R}\setminus\{x\})$. We now claim that $q(x) \in q(\mathbb{R}\setminus\{x\})$ is equivalent to $x \in \mathbb{R}\setminus\{x\}$. If this is the case, we got the contradiction. Let's see why this is the case. Clearly $x \in \mathbb{R}\setminus\{x\} \implies q(x) \in q(\mathbb{R}\setminus\{x\})$. On the other hand, suppose $q(x) \in \mathbb{R}\setminus\{x\}$. This means:

$$\exists l \in q(\mathbb{R}\setminus\{x\}) : q(x) = l.$$

We then have the following chain of equivalences:

$$q(x) \in q(\mathbb{R}\setminus\{x\}) \iff \exists l \in q(\mathbb{R}\setminus\{x\}) : q(x) = l \iff \exists n \in \mathbb{R}\setminus\{x\} : [n] = l \wedge q(x) = [n].$$

But this is equivalent to:

$$\exists s \in \mathbb{R}\setminus\{x\} : s = x \iff x \in \mathbb{R}\setminus\{x\},$$

which concludes that $q(x) \in q(\mathbb{R}\setminus\{x\}) \iff x \in \mathbb{R}\setminus\{x\}$. With this contradiction, we can conclude that $[a] \neq [x]$, which implies that $a \neq x$, that is, $a \in \mathbb{R}\setminus\{x\}$, as desired. Conversely, suppose $z \in \mathbb{R}\setminus\{x\}$. Then $q(z) \in q(\mathbb{R}\setminus\{x\})$. This means:

$$\exists m \in q(\mathbb{R}\setminus\{x\}) : q(z) = m.$$

Moreover:

$$\exists n \in \mathbb{R}\setminus\{x\} : [n] = m = q(z),$$

which is precisely the statement that there exists $[n] \in q(\mathbb{R}\setminus\{x\}) : [n] = q(z) = m$. But this is equivalent to:

$$z \in q^{-1}(q(\mathbb{R}\setminus\{x\})).$$

Hence, as desired:

$$q^{-1}(q(\mathbb{R}\setminus\{x\})) = \mathbb{R}\setminus\{x\}.$$

Having shown that $q^{-1}(q(\mathbb{R}\setminus\{x\})) = \mathbb{R}\setminus\{x\}$, we can proceed to conclude that $q(\mathbb{R}\setminus\{x\})$ is open in the quotient space, as it is the preimage of an open set under the continuous canonical projection. Namely, it is the image of $\mathbb{R}\setminus\{x\} = (-\infty, x) \cup (x, \infty)$. This is an open set, as it is the union of two open sets. Moreover, we have:

$$y \in \mathbb{R}, y \neq x \iff y \in \mathbb{R}\setminus\{x\} \implies q(y) \in q(\mathbb{R}\setminus\{x\}).$$

And clearly as $x \notin \mathbb{R} \setminus \{x\}$:

$$q(x) \notin q(\mathbb{R} \setminus \{x\}).$$

We thus have found an open set, which contains $q(y)$, but does not contain $q(x)$. This shows that $\mathbb{R}/\sim$ is T_0.

(b) $\mathbb{R}/\sim$ is not T_1, as $q^{-1}(\{q(0)\}) = \mathbb{Q}$. But we know that $\mathbb{Q}$ is not closed in $\mathbb{R}$, as it is not its own closure: in some more detail, there are rational sequences, that do not converge in $\mathbb{Q}$. This contradicts the fact that all singletons are closed, which should be the case for a T_1 space.

(c) T_2: As $T_2 \implies T_1$, we can immediately conclude $\mathbb{R}/\sim$ is not T_2.

(d) We have $q(\sqrt{2}) \notin q(\{\pi\})$ and $q(\{\pi\})$ is closed in $\mathbb{R}/\sim$. Let $U, V \subseteq \mathbb{R}/\sim$ be open neighbourhoods of $q(\sqrt{2}), q(\{\pi\})$ respectively. Then $q^{-1}(U)$ and $q^{-1}(V)$ are open non-empty subset of $\mathbb{R}$. Hence, by denseness, they both contain a rational number. Therefore, $q(0) \in U \cap V \neq \varnothing$. This proves that $\mathbb{R}/\sim$ is not T_3.

(e) T_4: The sets $q(\{\pi\})$ and $q(\{\sqrt{2}\})$ are closed in $\mathbb{R}/\sim$. Let $U, V \subseteq \mathbb{R}/\sim$ be open neighbourhoods in $\mathbb{R}/\sim$ of $q(\sqrt{2})$ and $q(\{\pi\})$ respectively. Then $q^{-1}(U)$ and $q^{-1}(V)$ are open non-empty subset of $\mathbb{R}$. As $\mathbb{R}$ is dense, both of them contain a rational number, which concludes $q(0) \in U \cap V \neq \varnothing$. This provse that $\mathbb{R}/\sim$ is not T_4.

6.3 Metrizable products

1. Show that a countable product of metrizable spaces is metrizable.

2. Is every product of metrizable spaces metrizable? Why (not)?

Solution 6.7.

1. We are given a countable collection of non-empty topological spaces, which are metrizable $\{X_n, \rho_n\}_{n=1}^{\infty}$. We denote their Cartesian product as

$$X := \prod_{n=1}^{\infty} X_n.$$

We define a metric $\rho : X \times X \to \mathbb{R}$ as:

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{\rho_n(x_n, y_n)}{2^n(1+\rho_n(x_n,y_n)}.$$

We claim that ρ induces the topology, which is equivalent to the product topology on $\mathbb{X}$. If this is true, then X is metrizable. Before we do this, let us simplify the metric, which can be done via the following lemma.

Lemma 6.8. *Let $\alpha_n = \frac{\rho_n}{1+\rho_n}$ and ρ_n be two metrics. Then the topology induced by α_n, ρ_n are equivalent.*

Proof. Let us introduce the following notation for open balls:

$$B_\varepsilon^\beta(x) := \{y|\beta(x,y) < \varepsilon\}.$$

Now, let us fix $\varepsilon > 0$. We will show that there exist $\delta_{\alpha_n}, \delta_{\rho_n} > 0$, such that:

$$B_{\delta_{\alpha_n}}^{\alpha_n}(x) \subseteq B_\varepsilon^{\rho_n}(x);\;\; B_{\delta_{\rho_n}}^{\rho_n}(x) \subseteq B_\varepsilon^{\alpha_n}(x).$$

However, before showing this, let us show why this is useful for the purpose of the proof. Suppose the above holds. Then, recall the definitions of the topologies induced by the metrics α_n, ρ_n:

$$A \in \mathcal{O}_{\rho_n} \iff \forall x \in A : \exists \varepsilon > 0 : B_\varepsilon^{\rho_n}(x) \subset A;$$

$$A \in \mathcal{O}_{\alpha_n} \iff \forall x \in A : \exists \varepsilon > 0 : B_\varepsilon^{\alpha_n}(x) \subset A.$$

Supposing we proved the statements above, we then have:

$$A \in \mathcal{O}_{\rho_n} \iff \forall x \in A : \exists \varepsilon > 0 : B_{\delta_{\alpha_n}}^{\alpha_n}(x) \subseteq B_\varepsilon^{\rho_n}(x) \subset A \implies A \in \mathcal{O}_{\alpha_n};$$

$$A \in \mathcal{O}_{\alpha_n} \iff \forall x \in A : \exists \varepsilon > 0 : B_{\delta_{\rho_n}}^{\rho_n}(x) \subseteq B_\varepsilon^{\alpha_n}(x) \subset A \implies A \in \mathcal{O}_{\rho_n}.$$

Which shows indeed that $\mathcal{O}_{\rho_n} = \mathcal{O}_{\alpha_n}$. Now let us prove the existence of $\delta_{\alpha_n}, \delta_{\rho_n}$:

$$m \in B_{\delta_{\alpha_n}}^{\alpha_n}(x) \iff \alpha(m,x) < \delta_{\alpha_n} \iff \frac{\rho_n(m,x)}{1+\rho_n(m,x)} < \delta_{\alpha_n} \iff \rho_n(m,x) < \delta_{\alpha_n} + \delta_{\alpha_n}\rho_n(m,x).$$

$$\rho_n(m,x) < \delta_{\alpha_n} + \delta_{\alpha_n}\rho_n(m,x) \iff \rho_n(m,x)(1-\delta_{\alpha_n}) < \delta_{\alpha_n} \iff \rho_n(m,x) < \frac{\delta_{\alpha_n}}{1-\delta_{\alpha_n}}.$$

Thus for the condition $m \in B_\varepsilon^{\rho_n}(x)$ to be satisfied, we have to choose δ_{α_n} such that:

$$\frac{\delta_{\alpha_n}}{1-\delta_{\alpha_n}} = \varepsilon \iff \delta_{\alpha_n} = \varepsilon - \varepsilon\delta_{\alpha_n} \iff \delta_{\alpha_n}(1+\varepsilon) = \varepsilon \iff \delta_{\alpha_n} = \frac{\varepsilon}{1+\varepsilon}.$$

This concludes the existence of δ_{α_n}. Now let us get to the existence of δ_{ρ_n}:

$$m \in B_{\delta_{\rho_n}}^{\rho_n}(x) \iff \rho_n(m,x) < \delta_{\rho_n}.$$

Note that $\rho_n(m,x) + 1 \geq 1$, which implies $\frac{\rho}{1+\rho} \leq \rho$. Hence, we can pick $\delta_{\rho_n} = \varepsilon$ and this does the job. This concludes the existence of δ_{ρ_n} and of the proof of the lemma. The two topologies coincide. $\square$

We therefore can work with the metric $\rho : X \times X \to \mathbb{R}$,

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{\rho_n(x_n,y_n)}{2^n}.$$

Let us show that this is indeed a metric, given $\rho_n(x,y)$ are:

(a) $\rho(x,y)=0 \iff x=y$. To see this, consider:

$$\rho(x,y)=\sum_{n=1}^{\infty}\frac{\rho_n(x_n,y_n)}{2^n}=0.$$

But we know that $\rho_n(x_n,y_n)=0 \iff x_n=y_n\forall n$. Thus $x=y$.

(b) Symmetry: $\rho(x,y)=\rho(y,x)$. To see this:

$$rho(x,y)=\sum_{n=1}^{\infty}\frac{\rho_n(x_n,y_n)}{2^n}=\sum_{n=1}^{\infty}\frac{\rho_n(y_n,x_n)}{2^n}=\rho(y,x),$$

where we used that $\rho_n(x_n,y_n)$ are symmetric, since they are metrics.

(c) Triangle inequality: $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$. To see this:

$$\rho(x,z)=\sum_{n=1}^{\infty}\frac{\rho_n(x_n,z_n)}{2^n}\leq\sum_{n=1}^{\infty}\frac{\rho_n(x_n,y_n)}{2^n}+\sum_{n=1}^{\infty}\frac{\rho_n(y_n,z_n)}{2^n}=\rho(x,y)+\rho(y,z),$$

where we used that each metric ρ_n satisfies the triangle inequality.

So far, we have a metric on the product. Moreover, this metric is bounded by 1, since:

$$\frac{\rho(x,y)}{\rho(x,y)+1}\leq 1 \iff \rho(x,y)\leq\rho(x,y)+1 \iff 0\leq 1.$$

We now show that the metric ρ induces the product topology on X. Let O be a basic open set in the product topology, so

$$O=\prod_{n\in\mathbb{N}}:\exists F\subset\mathbb{N}\text{ finite }:n\notin F\iff X_n=O_n.$$

We want to show that O is open in the topology induced by ρ. This is equivalent to finding an open ball around each $x\in O$, such that:

$$\forall x\in O:\exists r>0:B_r^{\rho}(x)\subset O.$$

Now, for $n\in F$, we have that $x_n\in O_n$, which is open in X_n, so we have $r_n>0:B_{r_n}^{\rho_n}(x_n)\subset O_n$ by the fact that the topology on X_n is induced by ρ_n. As we have finitely many r_n to consider, we can find $0<r<1$, such that $r\leq\frac{r_n}{2^n}\forall n\in F$. To see this, take $y\in B_r^{\rho}(x)$. This is equivalent to:

$$y\in B_r^{\rho}(x)\iff\rho(y,x)<r.$$

We know that:

$$\frac{\rho_n(x_n,y_n)}{2^n}\leq\rho(x,y)<r\leq\frac{r_n}{2^n},$$

which implies that for such $n\in F$, $y_n\in O_n$, and as the other $O_n=X_n$ by the form of the basic open set, we have that indeed $y\in O$. As y was arbitrary, we conclude $B_r^{\rho}(x)\subset O$.
We now do the other inclusion. We start with an open ball in the ρ-induced topology, and show that it is open in the product topology. To this end, start with $B_r^{\rho}(x)$ for some arbtirary

$x \in X, r > 0$. We want to show that $x \in O \subset B_r^\rho(x)$. To do this, pick $N \in \mathbb{N} : \frac{1}{2^N} < \frac{r}{2}$. For $1 \leq k \leq N$ consider the open balls

$$O_k := B^\rho_{\frac{r}{2N}}(x_k),$$

and for $k \geq N = +1$, we set $O_k = X_k$. We now claim that

$$O = \prod_{k \in \mathbb{N}} O_k \subset B_r^\rho(x).$$

Clearly, O is a basic open set in the product topology on X, moreover $x \in O$. To see this: let $y \in O$. Then for $k \leq N$, we have $\rho_k(x_k, y_k) < \frac{r}{2N}$, which implies:

$$\sum_{k=1}^{N} \frac{\rho_k(x_k, y_k)}{2^k} \leq \sum_{k=1}^{N} \rho_k(x_k, y_k) < N \cdot \frac{r}{2N} = \frac{r}{2}.$$

On the other hand:

$$\sum_{k=N+1}^{\infty} \frac{\rho_k(x_k, y_k)}{2^k} \leq \sum_{k=N=1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \frac{r}{2},$$

where we have used that ρ_k are bounded by 1, as was shown above. Moreover, we used our choice for N. Putting all together, we obtain for $y \in O$:

$$\rho(x, y) < \frac{r}{2} + \frac{r}{2} \iff \rho(x, y) < r,$$

as desired. This shows that the topology on X is induced by ρ, thus, X is metrizable.

2. Only countable products of non-singleton metrizable spaces can be metrizable. In particular, $\{0,1\}^{\mathbb{R}}$ for $\{0,1\}$ having the discrete topology is not metrizable. To see this, let $X := \prod_{i \in}$ have the product topology and let X_i be T_1 spaces that have at least two points. If X is metrizable, then it is first countable. So let $x := (x_i)_{i \in I} \in X$ and $(U_j)_{j \in \mathbb{N}}$ be an open neighbourhood basis for $x \in X$. For all $j \in \mathbb{N}$, there is a finite set $J_j \subseteq I$, such that $pr_i(U_j) = X_i$ for all $i \notin J_j$. Now, let $i \in I$. Then $x_i \in V_i := X_i \setminus \{y_i\}$ for some $y_i \in X_i \setminus \{x_i\}$. By continuity, $pr^{-1}(V_i)$ is an open neighbourhood of x and therefore $pr^{-1}(V_i)$ contains U_j for some $j \in \mathbb{N}$. Since $pr(U_j) \subseteq pr(V_i) = X_i \setminus \{y_i\} \neq X_i$, we have that $i \in J_j$. This proves that $I \subseteq \bigcup_{j \in \mathbb{N}} L_j$ is countable.

Remark 6.9. *Throughout the whole exercise, the axiom of choice had to be used, as we had to choose a metric on each topological space. If we want to be pedantic, we actually do not need the axiom of choice for uncountable collections. Countable choice is enough, as we have to choose only countably many metrics.*

7 Handout5

Proof. We want to show that the collection

$$G := \{F \cap U | F \in \mathcal{F}, U \in \mathcal{N}_x\}$$

is a filter. To this end, we check the three axioms:

1. G is non-empty, as x is a cluster point. Clearly, $X \in G$, as $\mathcal{F}$ is a filter, so $X \in \mathcal{F}$, and X is an open set containing x, hence $X \in \mathcal{N}_x$. We then have $X \cap X = X \in G$.

2. We want to show $U, V \in G \implies U \cap V \in G$. Let $U, V \in G$. Then:$U = P_1 \cap O_1$ and $V = P_2 \cap O_2$, where $P_1, P_2 \in \mathcal{F}, O_1, O_2 \in \mathcal{N}_x$. We then have:

$$U \cap V = (P_1 \cap O_1) \cap (P_2 \cap O_2) = (P_1 \cap P_2) \cap (O_1 \cap O_2)$$

   As $P_1, P_2 \in \mathcal{F}$, by second axiom of filter, we have that $P_1 \cap P_2 \in \mathcal{F}$ and similarly, the intersection of twoo neighbourhoods of the same point is again a neighbourhood of the same point. We conclude: $U \cap V \in G$, provided that $U, V \in G$.

3. Suppose $A \in G$. Then $A = P \cap U, P \in \mathcal{F}, U \in \mathcal{N}_x$. We now claim that if $P \cap U \subseteq V$, then $V \in \mathcal{F}$. To see this, we have to show that V can be written as $V = B \cap C$ for some $B \in \mathcal{F}, C \in \mathcal{N}_x$. Indeed, this is the case for $B = F \cup V$ and $C = U \cup V$. In the first place, $F \cup V \in \mathcal{F}$, since $F \in \mathcal{F}$ and $F \subset F \cup V$ and we apply axiom 3 of filter. Similarly, If $U \in \mathcal{N}_x$, any enlargement of it is a neighbourhood still. We now have to show what we said before that $V = B \cap C$. To this end, consider:

$$B \cap C = (F \cup V) \cap (U \cup V).$$

   We now apply $E \cap (K \cup L) = (E \cap K) \cup (E \cap L)$ for $E = F \cup V, K = U, L = V$. We then have:

$$(F \cup V) \cap (U \cup V) = ((F \cup V) \cap U) \cup ((F \cup V) \cap V).$$

   We now apply distributivity:

$$((F \cup V) \cap U)) \cup ((F \cup V) \cap V) = ((F \cap U) \cup (V \cap U)) \cup ((F \cap V) \cup (V \cap V)).$$

   We now see that $F \cap U \subseteq V$,by assumption. Clearly $V \cap U \subseteq V$ and $F \cap V \subseteq V$. Moreover $V \cap V = V$, hence the above is V, as desired. We thus have showed that:

$$V = (F \cup V) \cap (U \cup V),$$

   where $F \cup V \in \mathcal{F}$, $U \cup V \in \mathcal{N}_x$. Therefore, $V \in G$, as desired.

□

A De Morgan Laws

Lemma A.1. *Let $(V_i)_{i\in I} \subseteq X$ be an arbitrary collection of sets. Then:*

1. $X \setminus \left(\bigcup_{i\in I} V_i\right) = \bigcap_{i\in I} (X \setminus V_i)$;
2. $X \setminus \left(\bigcap_{i\in I} V_i\right) = \bigcup_{i\in I} (X \setminus V_i)$.

Proof. 1. $y \in \bigcup_{i\in I} V_i \iff \exists i \in I : y \in V_i$. Thus, the negation of this reads:

$$y \notin \bigcup_{i\in I} V_i \iff \forall i \in I : y \notin V_i.$$

Now using that all the $(V_i)_{i\in I}$ are subsets of X, we have:

$$y \in \left(X \setminus \left(\bigcup_{i\in I} V_i\right)\right) \iff \forall i \in I : y \in (X \setminus V_i).$$

But by the definition of intersection:

$$\forall i \in I : y \in (X \setminus V_i) \iff y \in \left(\bigcap_{i\in I} (X \setminus V_i)\right).$$

Thus, we have, as desired:

$$y \in \left(X \setminus \left(\bigcup_{i\in I} V_i\right)\right) \iff y \in \left(\bigcap_{i\in I} (X \setminus V_i)\right).$$

2. $y \in \bigcap_{i\in I} V_i \iff \forall i \in I : y \in V_i$. Thus, the negation of this reads:

$$y \notin \bigcap_{i\in I} V_i \iff \exists i \in I : y \notin V_i.$$

But since all the $(V_i)_{i\in I}$ are subsets of X, we have:

$$y \in \left(X \setminus \left(\bigcap_{i\in I} V_i\right)\right) \iff \exists i \in I : y \in (X \setminus V_i).$$

Using the definition of union on the RHS of the above equivalence gives the desired result:

$$y \in \left(X \setminus \left(\bigcap_{i\in I} V_i\right)\right) \iff y \in \left(\bigcup_{i\in I} (X \setminus V_i)\right).$$

$\square$

Corollary A.2. *Let $(V_i)_{i\in\mathbb{N}} \subseteq X$ be a finite collection of sets. Then:*

1. $X \setminus \left(\bigcup_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} (X \setminus V_i)$;
2. $X \setminus \left(\bigcap_{i=1}^{n} V_i\right) = \bigcup_{i=1}^{n} (X \setminus V_i)$.

Proof. Nearly trivial. Let I be a finite set of cardinality n. Then apply lemma A.1. □

B Injective, bijective functions

Lemma B.1. *Let X,Y be sets and $f : X \to Y$ be a bijective function. Then $g : X \to Z, g(x) := f(x)$ is injective for all $Z \supseteq Y$.*

Proof. Suppose g is not injective. That is:

$$\exists x,y \in X : g(x) = g(y) \not\Rightarrow x = y.$$

But using the definition of g, this translate to:

$$\exists x,y \in X : f(x) = f(y) \not\Rightarrow x = y,$$

which contradicts that f is bijective. Hence g has to be injective. □

Proposition B.2. *Let $f : A \to Y$ be a function and let $X \subseteq Y$. Then $A \setminus f^{-1}(X) = f^{-1}(Y \setminus X)$.*

Proof.

$$a \in f^{-1}(Y \setminus X) \iff f(a) \in Y \setminus X \iff f(a) \notin X \iff a \notin f^{-1}(X) \iff a \in A \setminus f^{-1}(X).$$

□

C The axiom of choice

Definition C.1. *Let $(H_i)_{i\in I}$ be an indexed family of sets by the index set I. Then, their cartesian product is defined as:*

$$\prod_{i\in I} H_i := \left\{ f : I \to \bigcup_{i\in I} H_i | f(i) \in H_i \forall i \in I \right\}.$$

To gain some familiarity with this abstract definition, let us write it out explicitly for finite products:

$$\prod_{i=1}^{n} H_i := \left\{ f : \{1,\dots,n\} \to \bigcup_{i=1}^{n} H_i | f(i) \in H_i, i = \overline{1,\dots,n} \right\}.$$

Let us see how it relates to our intuition, namely what we know about cartesian product of two sets:

$$\prod_{i=1}^{2} H_i := \left\{ f : \{1,2\} \to \bigcup_{i=1}^{2} H_i | f(i) \in H, i = 1,2 \right\}.$$

Recall, our intuition says:

$$H_1 \times H_2 := \{(h_1,h_2) | h_1 \in H_1, h_2 \in H_2\}.$$

Now let us show that our intuition is in bijective correspondence with the abstract definition. To this end, we define a map from $H_1 \times H_2$ to $\prod\limits_{i=1}^{2} H_i$.

$$\psi : H_1 \times H_2 \to \prod_{i=1}^{2} H_i, \;\; \psi((h_1,h_2)) = f : \;\; f(1) = h_1, \;\; f(2) = h_2.$$

Its inverse reads

$$\psi^{-1} : \prod_{i=1}^{2} H_i \to H_1 \times H_2, \;\; \psi^{-1}(f) = (f(1), f(2)).$$

They are indeed inverses of each other, as:

$$(\psi^{-1} \circ \psi)((h_1,h_2)) = \psi^{-1}(\psi((h_1,h_2))) = \psi^{-1}(f) = (f(1), f(2)) = (h_1,h_2) \implies \psi^{-1} \circ \psi = id_{H_1 \times H_2};$$

$$(\psi \circ \psi^{-1})(f) = \psi(\psi^{-1}(f)) = \psi(f(1), f(2)) = f \implies \psi \circ \psi^{-1} = id_{\prod\limits_{i=1}^{2} H_i}.$$

We now come to the axiom of choice, which we need, in order to guarantee that products of non-empty sets are non-empty.

Axiom C.2. *Let $(H_i)_{i \in I}$ be an indexed family of non-empty sets, indexed by an index set I. Then $\prod\limits_{i \in I} H_i \neq \varnothing$.*

Definition C.3. *The projection to the i-th factor is a map defined as*

$$pr_i : \prod_{j \in I} X_j \to X_i, \;\; pr_i(f) := f(i)$$

Remark C.4. *The above is well-defined, as the definition of the product assumed that $f(i) \in X_i$.*

Theorem C.5. *Suppose the axiom of choice holds and $(X_i)_{i \in I}$ is an indexed family of non-empty sets. Then, the projection maps $pr_i : \prod\limits_{j \in I} X_j \to X_i$ are surjective.*

Before proving this theorem, let us consider a concrete example, to get familiar with how this abstract definition of products works, and why the projections are surjective. To this end, consider:

$$I = \{0,1,2\}, X_0 = \{0,1,2,3\}, X_1 = \{a,b\}, X_2 = \{a,0\}.$$

Then, the product space:

$$\prod_{j=0}^{2} X_j = \{f : \{0,1,2\} \to \{0,1,2,3,a,b\} | f(j) \in X_j \text{ for all } j \in I\}.$$

The axiom of choice tells us that there exists $F \in \prod_{j=0}^{2} X_j$. Without loss of generality, we can assume that

$$F(0) = 0, F(1) = a, F(2) = a.$$

We then have:

$$pr_i(F) = \begin{cases} F(0) = 0 & \text{if } i = 0, \\ F(1) = a & \text{if } i = 1, \\ F(2) = a & \text{if } i = 2. \end{cases}$$

Now, by existence of this F, we claim that we can construt all other elements, which are needed to make pr_i surjective. We do this procedure for pr_0 first. Since F exists and $1 \in X_0$, we can define:

$$G : \{0,1,2\} \to \{0,1,2,3,a,b\}, G(i) = \begin{cases} 1 & \text{if } i = 0, \\ F(i) & \text{if } i = 1,2. \end{cases}$$

Clearly, G is well defined, as $G(i) = F(i) \in X_i$ if $i = 1,2$ and $F(i)$ was well-defined by the axiom of choice. Moreover $G(0) = 1, 1 \in X_0$. We then have:

$$pr_0(G) = G(0) = 1.$$

So far we have that the existence of F allowed us to construct a function G, so that $pr_0(G) = 1$. Hence, we have $pr_0(F) = 0, pr_0(G) = 1$. We still need to find functions $H, K \in \prod_{j=0}^{2} X_j$ using only F, so that $pr_0(H) = 2, pr_0(L) = 3$, if we want pr_0 to be surjective. We can do this, since we can define:

$$H : \{0,1,2\} \to \{0,1,2,3,a,b\}, H(i) = \begin{cases} 2 & \text{if } i = 0, \\ F(i) & \text{if } i = 1,2. \end{cases}$$

Clearly, H is well-defined, as F was and $H(0) = 2 \in X_0$. Thus:

$$pr_0(H) = H(0) = 2.$$

Similarly, we define K as:

$$K : \{0,1,2\} \to \{0,1,2,3,a,b\}, H(i) = \begin{cases} 3 & \text{if } i = 0, \\ F(i) & \text{if } i = 1,2. \end{cases}$$

Clearly, K is well-defined as F was and $K(0) = 3 \in X_0$. Moreover:

$$pr_0(K) = K(0) = 3.$$

We conclude, that we could make pr_0 surjective, since:

$$\forall y \in X_0 : \exists M \in \prod_{j=0}^{2} X_j : pr_i(M) = y.$$

Or in more detail:

$$3 \in X_0,\ K \in \prod_{j=0}^{2} X_j : pr_0(K) = 3;$$

$$2 \in X_0,\ H \in \prod_{j=0}^{2} X_j : pr_0(H) = 2;$$

$$1 \in X_0,\ G \in \prod_{j=0}^{2} X_j : pr_0(G) = 1;$$

$$0 \in X_0,\ F \in \prod_{j=0}^{2} X_j : pr_0(F) = 0.$$

This can be compactly written as:

$$pr_0(\{F, G, H, K\}) = X_0.$$

Similarly, by the existence of F, we can make pr_1 surjective aswell. So far, we have that:

$$pr_1(F) = F(1) = a.$$

Now we can define L as:

$$L : \{0,1,2\} \to \{0,1,2,3,a,b\}, L(i) = \begin{cases} F(i) & \text{if } i = 0,2; \\ b & \text{if } i = 1 \end{cases}.$$

Clearly, L is well-defined as F is and $L(1) = b \in X_1$. Moreover:

$$pr_1(L) = L(1) = b.$$

Thus, we found $F, L \in \prod_{j=0}^{2} X_j$, which make pr_1 surjective. That is:

$$pr_1(\{F, L\}) = \{a, b\} = X_1.$$

Finally, to make pr_2 surjective, define N as:

$$N : \{0,1,2\} \to \{0,1,2,3,a,b\}, N(i) = \begin{cases} F(i) & \text{if } i = 0,1; \\ 0 & \text{if } i = 2. \end{cases}$$

Clearly, N is well-defined as F is and $N(2) = 0 \in X_2$. Moreover:

$$pr_2(N) = N(2) = 0.$$

Thus, we found $F, N \in \prod_{j=0}^{2} X_j$, which make pr_2 surjective, namely:

$$pr_2(\{F,N\}) = \{a, 0\} = X_2.$$

We conclude, that having the existence of F allowed us to construct all other elements of $\prod_{j=0}^{2} X_j$ needed, in order to make all pr_0, pr_1, pr_2 surjective. This is what we will do in the more general case. Let us thus start the proof of theorem $C.5$

Proof. Let $a \in X_i$ be arbitrary. We are allowed to do this, as X_i are non-empty by assumption. Then, by the axiom of choice, we have that there exists $f \in \prod_{j \in I} X_j$, i.e. there exists a function:

$$f : I \to \bigcup_{i \in I} X_i, \; f(i) \in X_i \;\; \forall i \in I.$$

We can now define a function g as:

$$g : I \to \bigcup_{i \in I} X_i, \; g(j) = \begin{cases} f(j) & \text{if } j \in I, j \neq i; \\ a & \text{if } i = j. \end{cases}$$

The thus defined function g is well-defined, that is $g \in \prod_{j \in I} X_j$, since $g(j) = f(j)$ is in X_j for all $j \in I, j \neq i$ and $g(i) = a \in X_i$ by assumption. Moreover, we have:

$$pr_i(g) = g(i) = a.$$

Since $a \in X_i$ was arbitrary, this concludes that we can always find $g \in \prod_{j \in I} X_j$, so that $pr_i(g) = a$, that is: pr_i is surjective for all $i \in I$. $\square$

We can say even more, the converse holds aswell:

Proposition C.6. *Let $(X_i)_{i \in I}$ be an indexed family of non-empty sets indexed by I. If the projections to the i-th component $pr_i : \prod_{j \in I} X_j \to X_i$ are surjective for all $i \in I$, then the axiom of choice holds.*

Proof. Suppose the projections pr_i are surjective. Then for each $y_i \in X_i$, there exists $f \in \prod_{j\in I} X_j$, such that $pr_i(f) = y$. As X_i is non-empty, there must exist one element in $\prod_{j\in I} X_j$, hence the axiom of choice holds, as desired. □

Hence, we can formulate in a compact way what we showed:

Theorem C.7. *Let $(X_i)_{i\in I}$ be an indexed family of sets indexed by I. Then the axiom of choice is equivalent to the projection maps pr_i being surjective for all $i \in I$.*

There is a technical remark: the index set need not be non-empty, but each of the sets in the collection need be non-empty. This follows from the fact that the function $f : \varnothing \to \varnothing$ is an element of the product always,so the product is non-empty. In some more details, the function $f : \varnothing \to \varnothing$ is a relation on $\varnothing \times \varnothing := \{(a,b) | a \in \varnothing, b \in \varnothing\} = \varnothing$, where the last equality holds as there are no elements in the empty set. So the product indexed by an empty set of non-empty sets is always non-empty, contains this one function $f : \varnothing \to \varnothing$. Moreover, there is another statement, which follows by the same logic as before, but not related to the axiom of choice: the arbitrary non-empty product of empty sets is empty. This has nothing to do with the axiom of choice, as there the elements in the product are assumed to be non-empty.

D Equivalence relations

Definition D.1. *A binary relation $\sim$ on a set X is called an equivalence relation, if it is:*

1. *reflexive: $\forall x \in X : x \sim x$;*
2. *symmetric: $\forall x \in X : x \sim y \implies y \sim x$;*
3. *transitive: $\forall x, y, z \in X : x \sim y, y \sim z \implies x \sim z$.*

The tuple $(X, \sim)$ is called a setoid.

Terminology: If $x \sim y$, x and y are said to be related.

Remark D.2. *Several sources usually require for symmetry $\iff$ instead of $\implies$. However, the two are equivalent, because it holds for all $x, y \in X$. Letting $x' = y, y' = x$, we obtain the converse in our def, so it is $\iff$.*

Definition D.3. *Let $(X, \sim)$ be a setoid and $a \in X$. The equivalence class of a under $\sim$ is the set*

$$[a] := \{x \in X | x \sim a\}.$$

The elements of an equivalence class $y \in [a]$ are called representatives.

Definition D.4. *Let $(X, \sim)$ be a setoid. Then the quotient set $X/\sim$ induced by $\sim$ is the set of equivalence classes, i.e.*

$$X/\sim := \{[x] \mid x \in X\}.$$

Definition D.5. *Let X be a set. A set $S \subset \mathcal{P}(X)$ is called a partition if:*

1. $\forall X_1, X_2 \in S : X_1 \neq X_2 \implies X_1 \cap X_2 = \varnothing$;
2. $\bigcup_{X_i \in X} X_i = X$;
3. $\forall T \in S : T \neq \varnothing$.

Theorem D.6. *Let $(X, \sim)$ be a setoid. Then the quotient set $X/\sim$ forms a partition of X.*

Proof. We check that each condition of being a partition is satisfied. We start backwards:

1. Each equivalence class is non-empty, as $[x]$ contains at least x, so is non-empty. This clearly holds, as $x \sim x$ by the reflexivity.

2. We show that the union of all the equivalence classes is the whole set. We now claim that:

   $$\bigcup_{x \in X} [x] = X.$$

   To see that these two sets are equal, we prove both inclusions:

   (a) $X \subseteq \bigcup_{x \in X} [x]$:let $y \in X$. Then, clearly y is in some equivalence class $[x]$, namely in $[y]$, which holds by reflexivity.

   (b) $\bigcup_{x \in X} [x] \subseteq X$: let $y \in \bigcup_{x \in X} [x]$. Then y is in $[x]$ for some $x \in X$. Since equivalence classes are subsets of X, we have that $y \in X$.

3. Finally, we show that equivalence classes are disjoint:

   $$[X_1] \neq [X_2] \implies [X_1] \cap [X_2] = \varnothing.$$

   We will prove this by contrapositivity. Suppose the two equivalence classes are not disjoint, that is:

   $$[X_1] \cap [X_2] \neq \varnothing.$$

   We then have the chain of equivalences:

   $$\begin{aligned} [X_1] \cap [X_2] \neq \varnothing &\iff \exists z : z \in [X_1] \cap [X_2] \iff \exists z : z \in [X_1] \wedge z \in [X_2] \iff \exists z : z \sim X_1 \wedge z \sim X_2 \\ &\iff \exists z : X_1 \sim z \wedge z \sim X_2 \iff [X_1] = [X_2] \end{aligned}$$

   Thus, we showed that if $[X_1] \cap [X_2] \neq \varnothing$, then $[X_1] = [X_2]$. Clearly, then $[X_1] \cap [X_2] \neq \varnothing$. This finishes the proof by contrapositive.

□

Lemma D.7. *Let $(X, \sim)$ be a setoid. Then two equivalence classes are equal, if their representatives are related, i.e. $[x_1] = [x_2] \iff x_1 \sim x_2$.*

Proof. "$\Rightarrow$": Suppose $[x_1] = [x_2]$. Then $x_2 \in [x_1]$, and by definition of $[x_1]$, we have that $x_1 \sim x_2$. "$\Leftarrow$": Suppose $x_1 \sim x_2$. Let $c \in [x_2]$. Then, clearly $x_2 \sim c$. By transitivity, we also have:

$$x_1 \sim x_2, x_2 \sim c \implies x_1 \sim c.$$

Hence $c \in [x_1]$, that is $[x_2] \subseteq [x_1]$. On the other hand, let $d \in [x_1]$. Then, clearly $x_1 \sim d$. By symmetry, we have $d \sim x_1$. Using transitivity:

$$d \sim x_1, x_1 \sim x_2 \implies d \sim x_2.$$

Finally, using symmetry $x_2 \sim d$, which implies $d \in [x_2]$. Thus, $[x_1] \subseteq x_2$. Hence, $[x_1] = [x_2]$, as desired. □

Lemma D.8. *Let $\sim_X, \sim_Y, \sim_Z$ be equivalence relations on sets X, Y, Z. Let $f : X \to Y, g : Y \to Z$ be set maps inducing well defined maps on the equivalence classes:*

$$\tilde{f} : X/_{\sim_X} \to Y/_{\sim_Y}, \ \tilde{f}([x]_X) := [f(x)]_Y;$$

$$\tilde{g} : Y/_{\sim_Y} \to Z/_{\sim_Z}, \ \tilde{g}([y]_Y) := [g(y)]_Z.$$

Then $g \circ f : X \to Z$ induces a well-defined map $\widetilde{g \circ f} : X/_{\sim_X} \to Z/_{\sim_Z}$, $\widetilde{g \circ f}([x]_X) := [(g \circ f)(x)]_Z$ on the equivalence classes. Moreover, $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

Proof. $\tilde{f}, \tilde{g}$ are well defined, that is:

$$\tilde{f}([x_1]_X) = \tilde{f}([x_2]_X) \text{ if } x_1 \sim_X x_2; \ \tilde{g}([y_1]_Y) = \tilde{g}([y_2]_Y) \text{ if } y_1 \sim_Y y_2.$$

This can be rewritten as:

$$x_1 \sim_X x_2 \implies [f(x_1)]_Y = [f(x_2)]_Y, \ y_1 \sim_Y y_2 \implies [g(y_1)]_Z = [g(y_2)]_Z.$$

Or using lemma D.7:

$$x_1 \sim_X x_2 \implies f(x_1) \sim_Y f(x_2), \ y_1 \sim_Y y_2 \implies g(y_1) \sim_Z g(y_2).$$

We now want to show that $\widetilde{g \circ f}$ is well-defined:

$$x_1 \sim_X x_2 \implies (g \circ f)(x_1) \sim_Z (g \circ f)(x_2).$$

To see this, consider:

$$(g \circ f)(x_1) = g(f(x_1)).$$

Since $x_1 \sim_X x_2$, we have that $f(x_1) \sim_Y f(x_2)$, by well-definedness of $\tilde{f}$. Denote $y_1 := f(x_1), y_2 := f(x_2)$. We then have:

$$g(f(x_1)) = g(y_1).$$

Since $y_1 \sim_Y y_2$, using the well-definedness of $\tilde{g}$:

$$g(y_1) \sim_Z g(y_2).$$

Unravelling the definitions of y_1, y_2:

$$g(f(x_1)) \sim_Z g(f(x_2)).$$

Or equivalently:

$$(g \circ f)(x_1) \sim_Z (g \circ f)(x_2),$$

which shows that $\widetilde{g \circ f}$ is well-defined. Moreover, we have:

$$\widetilde{g \circ f}([x]_X) = [(g \circ f)(x)]_Z = [g(f(x))]_Z = \tilde{g}([f(x)]_Y) = \tilde{g}(\tilde{f}([x]_X)) = (\tilde{g} \circ \tilde{f})([x]_X).$$

Hence, we showed:

$$\widetilde{g \circ f}([x]_X) = (\tilde{g} \circ \tilde{f})([x]_X) \implies \widetilde{g \circ f} = \tilde{g} \circ \tilde{f},$$

which finishes the proof. □

D.1 Set theoretic quotients

In the previous chapter, we defined quotient sets. We now want to show that the quotient set satisfies a universal property.

Theorem D.9. *Let $(X, \sim)$ be a setoid and define $p : X \to X/\sim, p(x) := [x]$. Then:*

1. *The fibers of p are the equivalence classes of $\sim$ and p is surjective.*
2. *If $f : X \to Y$ is a map constant on the equivalence classes of $\sim$, there exists a unique induced map $\tilde{f} : X/\sim \to Y$, such that $f = \tilde{f} \circ p$.*

Proof. We will do the proof in two steps, as was stated in the theorem.

1. First, let us show that the fibers of p are the equivalence classes of $\sim$. To this end, let $[y] \in X/\sim$ be arbitrary:

$$p^{-1}([y]) = \{z \in X | p(z) = [y]\} = \{z \in X : [z] = [y]\} = \{z \in X : z \sim y\} = [y],$$

where in the second equality we used lemma D.7 and in the last equality we used the definition of equivalence class. This concludes that the fibers of p are equivalence classes of $\sim$. We now show that p is surjective:

$$\forall [y] \in X/\sim : \exists x \in X : p(x) = [y].$$

By definition of p, we have: $[x]=[y]$. So for all $[y]\in X/\sim$, we have to find $x\in X$, such that $[x]=[y]$. By lemma D.7 this is equivalent to finding $x\in X: x\sim y$. Now, by reflexivity, we have by setting $y=x$ that $y\sim y$, which concludes surjectivity.

2. We will divide this proof aswell into two parts: showing that there exists $\tilde{f}$, which satisfies the property, and then in the second part we will show that it is unique. First, we have to show that there exists $\tilde{f}: X/\sim\to Y$, such that $f=\tilde{f}\circ p$. To show this, we define $\tilde{f}$ as:

$$\tilde{f}: X/\sim\to Y,\ \ \tilde{f}([x]) := f(x).$$

$\tilde{f}$ from above is well-defined, which follows from f being constant on equivalence classes:

$$\tilde{f}([x]) = f(x) = f(y) = \tilde{f}([y]) \text{ if } x\sim y \iff [x]=[y].$$

Moreover, $\tilde{f}$ defined as such satisfies the property $f=\tilde{f}\circ p$, since:

$$(\tilde{f}\circ p)(x) = \tilde{f}(p(x)) = \tilde{f}([x]) = f(x) \implies \tilde{f}\circ p = f.$$

We now come to uniqueness. Suppose $g: X/\sim\to Y$ is a map, which satisfies $f=g\circ p$. We then have:

$$f(x) = (g\circ p)(x) = g(p(x)) = g([x]).$$

However, recall that we showed $\tilde{f}$ was defined:

$$\tilde{f}([x]) = f(x) \implies \tilde{f}([x]) = g([x]) \implies \tilde{f} = g,$$

which concludes the uniqueness proof.

$\square$

Remark D.10. *We see that the constancy of f on equivalence classes is a requirement, which is needed in order for $\tilde{f}$ to be well-defined as a map on the set-theoretic level.*

A nice counterexample is the following: consider $X=\mathbb{R}$, and $x\sim y \iff y-x\in\mathbb{Z}$. Now take $Y=\mathbb{S}^1$. We could define $\tilde{f}$ as:

$$\tilde{f}: \mathbb{R}/\sim\to\mathbb{S}^1,\ \ \tilde{f}([x]) := e^{i\pi x}.$$

However, this map won't be well-defined, as $0\sim 1 \iff [0]=[1]$, but $\tilde{f}([0])=1, \tilde{f}([1])=e^{i\pi}=-1$. We can fix this by defining $\tilde{f}$ as:

$$\tilde{f}: \mathbb{R}/\sim\to\mathbb{S}^1,\ \ \tilde{f}([x]) := e^{4\pi ix}.$$

This fixes the issue, as if $x\sim y$, we have that $x-y\in\mathbb{Z}$, which implies:

$$\tilde{f}([x]) = e^{4\pi ix} = 1\cdot e^{4\pi ix} = e^{4\pi i(y-x)}e^{4\pi ix} = e^{4\pi iy} = \tilde{f}([y]).$$