CHAPTER 2 MATRIX ALGEBRA

2.1 Matrices

A rectangular matrix $[A]$ of order $m \times n$ is

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mm} \end{bmatrix}$$

where a_{ij} is the element at ith row and jth column.

If $m=n$, it is a square matrix

If $m=1$, it is a row matrix

If $n=1$, it is a column matrix.

2.2 Row and column matrices

Row matrix $\lfloor A \rfloor = \lfloor a_{11} \quad a_{12} \quad a_{13} \quad a_{14} \cdots\cdots a_{1n}$

Column matrix $\{A\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{Bmatrix}$

2.3 Addition and subtration of matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

Commutative Law $\quad [A] - [B] = -[B] + [A]$

Associative Law $\quad \left([A] + [B]\right) - [C] = [A] + \left([B] - [C]\right)$

2.4 Scalar multipliers

A matix of order one by one is a scalar. Scalar is a quantity.

Let k be a scalar (a number), then

$$k[A] = \begin{bmatrix} ka_{11} & ka_{12} & \cdots\cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots\cdots & ka_{2n} \\ ka_{31} & & & \\ \vdots & & & \\ ka_{n1} & ka_{n2} & \cdots\cdots & ka_{nn} \end{bmatrix}$$

Example:

If $[A] = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$ $[B] = \begin{bmatrix} 2 & -4 \\ 6 & 8 \end{bmatrix}$ $[C] = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

Then

$$2[A] + 3[B] - 4[C] = \begin{bmatrix} 4 & 2 \\ 24 & 30 \end{bmatrix} = 2\begin{bmatrix} 2 & 1 \\ 12 & 15 \end{bmatrix}$$

2.5 Matrix multiplication

$\underset{2x2}{[C]} = \underset{2x3}{[A]} \underset{3x2}{[B]}$ can be written in detail as

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

where

$$\underset{2x2}{[C]} = \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}) & (a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}) \\ (a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}) & (a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}) \end{bmatrix}$$

If matrix [A] is of order mxp and matrix [B] is of order pxn and matrix [C] is the product of matrices [A] and [B], matrix [C] is then of order mxn.

The term c_{ij} is obtained by the formula that

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \;\cdots\cdots\; + a_{ip}b_{pj}$$

2.6 Fortran Statements for Matrix Multiplication

For $\underset{M\times N}{[C]} = \underset{M\times L}{[A]} \times \underset{L\times N}{[B]}$ we write

```
      DIMENSION A(100,100), B(100,100), C(100,100)
      input matrices [A] and [B]

      DO 10 I = 1,M
      DO 10 J = 1,N
      C(I,J)=0.0
      DO 10 K = 1,L
   10 C(I,J) = C(I,J) + A(I,K)*B(K,J)
```

Diagram Explanation of Matrix Multiplication

If $[A] = \begin{bmatrix} 1 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ $\quad [B] = \begin{bmatrix} 4 & 2 & 1 & 7 & 4 \\ 1 & 2 & 2 & 2 & 1 \\ 0 & 6 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 & 1 \\ 0 & 2 & 1 & 6 & 4 \\ 3 & 0 & 1 & 2 & 2 \end{bmatrix}$

and $\underset{3\times 5}{[C]} = \underset{3\times 6}{[A]} \times \underset{6\times 5}{[B]}$

we can write the multiplication procedure in the following diagram form:

$$\begin{array}{cc} & [B] \\ [A] & [C] \end{array}$$

$$\begin{array}{cc} & \begin{bmatrix} 4 & 2 & 1 & 7 & 4 \\ 1 & 2 & 2 & 2 & 1 \\ 0 & 6 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 & 1 \\ 0 & 2 & 1 & 6 & 4 \\ 3 & 0 & 1 & 2 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & 13 & 5 & 15 & 11 \\ 8 & 4 & 11 & \boxed{10} & 5 \\ 7 & 11 & 10 & 13 & 8 \end{bmatrix} \end{array}$$

For example; $c_{24} = \sum_{k=1}^{6} a_{2k} b_{k4} = 10$

Some notes for matrix multiplication

i Matrices are in general not commutative in multiplication:

$$[A][B] \neq [B][A]$$

Example: $[A][B] = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix}$

$[B][A] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & 10 \end{bmatrix}$ unequal

ii Two matrices [A] and [B] can only be multiplied together if they are conformable, i.e.,

"The number of columns in matrix [A] must be equal to the number of rows in matrix [B]"

$$\underset{M\times P}{[A]} \times \underset{L\times N}{[B]} = \underset{M\times N}{[C]} \quad \text{if and only if } P = L.$$

$$\underset{M\times P}{[A]}\ \underset{L\times Q}{[B]}\ \underset{R\times N}{[C]} = \underset{M\times N}{[D]} \quad \text{if and only if } Q = R \text{ and } P = L.$$

where [C] is premultiplied by [B] and [B] is premultiplied by [A]. [A] is postmultiplied by [B].

iii The matrices are associative in multiplication

$$[A][B][C] = ([A][B])[C] = [A]([B][C])$$

Example: $[A] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ $[B] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$[C] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$([A][B])[C] = \begin{bmatrix} 2 & -1 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}$$

$$[A]([B][C]) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}$$

2.7 <u>Transpose of Matrix</u>

Interchange the rows and columns of a matrix [A], we obtain the transposed matrix $[A]^T$.

$$([A]^T)^T = [A]$$

$$[A]^T + [B]^T = ([A] + [B])^T$$

If $[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ then $[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$

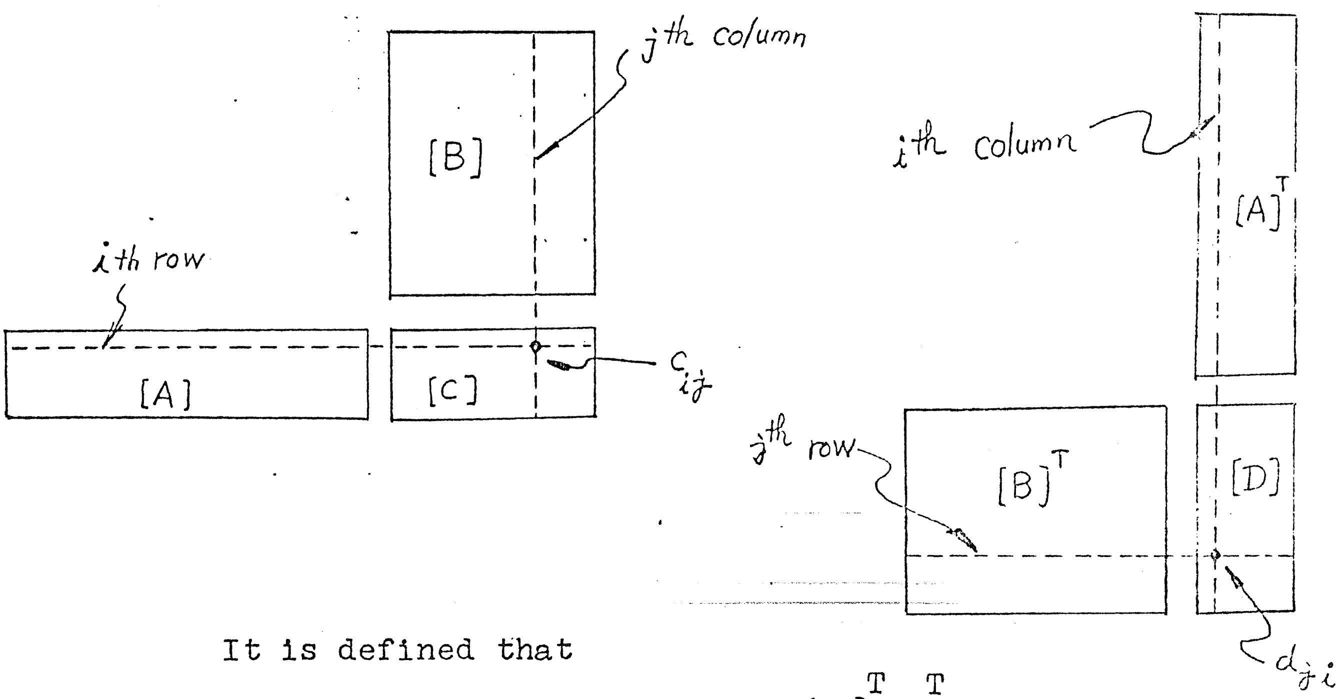
For $\underset{M\times N}{[A]}$ and $\underset{N\times L}{[B]}$, $[A][B]$ are conformable

But for $\underset{N\times M}{[A]^T}$ and $\underset{L\times N}{[B]^T}$, $\begin{cases} [B]^T[A]^T & \text{are conformable} \\ [A]^T[B]^T & \text{are not conformable.} \end{cases}$

Hence we know that

$$([A][B])^T \neq [A]^T[B]^T$$

<u>Derivation of the formula that $([A][B])^T = [B]^T[A]^T$</u>



It is defined that

$$[C] = [A][B] \text{ and } [D] = [B]^T[A]^T$$

From the figure it is seen that $c_{ij} = d_{ji}$, or in general $[D] = [C]^T$

It is therefore shown that

$$([A][B])^T = [B]^T[A]^T$$

the equation can be generalized as that

$$([A][B][C][D])^T = [D]^T[C]^T[B]^T[A]^T$$

Example: If $[A] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $[B] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

and if $[C] = [A][B]$, find $[C]^T$

We have two solutions for this problem.

Solution 1 $\quad [C] = [A][B] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1 & 2 & 1 \\ 3 & 3 & 0 & -1 \end{bmatrix}$$

Therefore

$$[C]^T = \begin{bmatrix} 3 & 3 \\ 1 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

Solution 2

$$[C]^T = [B]^T[A]^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 1 & 3 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

2.8 Special Matrices

(a) Square Matrix: If m = n for $[A]_{m \times n}$, then $[A]$ is a square matrix.

For symmetrical matrixm $a_{ij} = a_{ji}$ and $[A] = [A]^T$

For antisymmetrical matrix, $a_{ij} = -a_{ji}$ and $[A] = -[A]^T$

(b) Null Matrix

$a_{ij} = 0$ for all i and j.

(c) Diagonal Matrix (restricted to square matrix)

$a_{ij} = 0$ for $i \neq j$ and $a_{ij} \neq 0$ for $i = j$.

Thus a diagonal matrix is in the form that

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

If $[A]$ is a diagonal matrix and $[A]\{x\} = \{C\}$, what is the solution for the unknown vector $\{x\}$?

(d) <u>Unit Matrix</u> (Identity Matrix) $[I]$

$$a_{ij} = 0 \quad \text{for} \quad i \neq j, \quad a_{ij} = 1 \quad \text{for} \quad i = j$$

$$[I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[A][I] = [A], \quad [I][A] = [A]$$

(e) <u>Scalar Matrix</u>:

Scalar matrix is a special form of diagonal matrix. It is a diagonal matrix when $a_{ii} = a_{jj} = a_{11} = a_{22} = a_{33} = a_{44} = \ldots\ldots\ldots = \text{constant}$.

For example,

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} = 4\,[I]$$

(f) <u>Triangular Matrix</u> (Restricted to Square Matrix)

Upper triangular matrix: $a_{ij} = 0$ for i greater than j.

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & 2 & 9 & 0 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lower triangular matrix: $a_{ij} = 0$ for i smaller than j.

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 3 & 9 & 0 \\ 2 & 3 & 7 & 1 \end{bmatrix}$$

2.9 Matrix Partition (Submatrices)

The array of elements in a matrix may be divided into smaller arrays by horizontal and vertical dash lines. Such matrix is then refered to as a partitioned matrix, and the smaller arrays are called submatrices.

$$[A] = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where

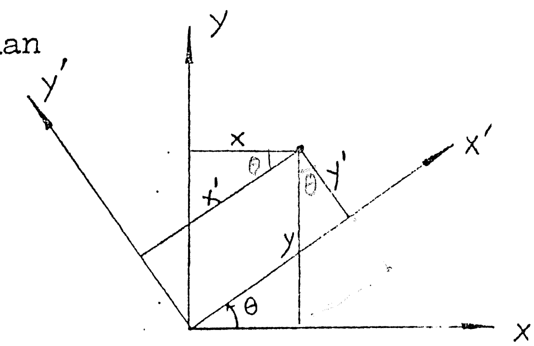
$$[A_{11}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; \quad [A_{12}] = \begin{Bmatrix} a_{13} \\ a_{23} \end{Bmatrix}$$

$$\lfloor A_{21} \rfloor = \lfloor a_{31} \quad a_{32} \rfloor ; \quad \text{and} \quad A_{22} = a_{33} .$$

2.11 Coordinate Transformation Matrix [T]

For example, the transformation matrix for coordinate transformation in a plan can be derived as follows.

Assuming θ is an angle that the coordinates (x, y) rotate, then



$$x' = x \cos\theta + y \sin\theta$$

$$y' = -x \sin\theta + y \cos\theta$$

or

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = [T] \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Example 2.4 3-D Coordinate Transformation

2.10 Orthogonal Matrix (restricted to square matrix)

If $[A]^{-1} = [A]^T$, matrix [A] is an orthogonal matrix.

Advantage: If matrix [A] is orthogonal, $[A]^{-1}$ can be obtained simply by transposing [A] without lengthy calculation for inverting.

If $[A]^{-1} = [A]^T$, then $[A][A]^T = [I]$

Assuming that $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

the property of orthogonality implies that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying out, we obtain

$$\begin{cases} a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \qquad \begin{cases} a_{11}\, a_{21} + a_{12}\, a_{22} + a_{13}\, a_{23} = 0 \\ a_{21}\, a_{31} + a_{22}\, a_{32} + a_{23}\, a_{33} = 0 \\ a_{31}\, a_{11} + a_{32}\, a_{12} + a_{33}\, a_{13} = 0 \end{cases}$$

Thus we reach the following conclusion:

<u>Conditions of Orthogonality</u>

1. The sum of the squares of each term in a row must be equal to unity.

2. The sum of the products of each pair of corresponding terms in two rows must be equal to zero.

Example: Show that the transformation matrix derived for coordinates rotation in the previous section is an orthogonal matrix. $[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

We check the matrix by the two conditions of orthogonality.

1 Summation of Squares $\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ (-\sin\theta)^2 + \cos^2\theta = 1 \end{cases}$ O.K.

2 Summation of products

$$-\sin\theta\cos\theta + \sin\theta\cos\theta = 0 \qquad \text{O.K.}$$

We thus conclude that the above coordinate transformation matrix is an orthogonal matrix.

2.12 Determinant (restricted to square matrix)

A determinant of a square matrix $[A]_{m \times m}$ is a scalar value function denoted by

$$\det[A] = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

2.12.1 Minor and Cofactor

The first minor of a determinant $\det[A]$, corresponding to the element a_{ij}, is defined as the determinant obtained by ommision of the *i*th row and *j*th column of $\det[A]$. Let us denote this minor by $\overline{M}_{ij}$.

Example: If $|A| = \begin{vmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 2 \\ 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 & 3 \end{vmatrix}$

the first minor for a_{32} is defined as

$$\overline{M}_{32} = \begin{vmatrix} 1 & 5 & 7 & 9 \\ 2 & 6 & 8 & 2 \\ 2 & 2 & 1 & 2 \\ 3 & 3 & 4 & 3 \end{vmatrix}$$

If the first minor $\overline{M}_{ij}$ is multiplied by $(-1)^{i+j}$, it becomes the cofactor of the term a_{ij}. Thus

$$\overline{A}_{ij} = (-1)^{i+j}\, \overline{M}_{ij}$$

Example:

$$\overline{A}_{32} = (-1)^{3+2}\, \overline{M}_{ij} = -\begin{vmatrix} 1 & 5 & 7 & 9 \\ 2 & 6 & 8 & 2 \\ 2 & 2 & 1 & 2 \\ 3 & 3 & 4 & 3 \end{vmatrix}$$

2.12.2 Expension of Determinant (Laplace Expansion Formula)

The determinant of a matrix $[A]$ can be found by the repetitive use of the Laplace Expansion formula:

$$\det \underset{m\times m}{[A]} = \sum_{k=1}^{m} a_{ik}\,\bar{A}_{ik} \quad \text{(i can be any row and i is not to be summed)}$$

or

$$\det \underset{m\times m}{[A]} = \sum_{k=1}^{m} a_{kj}\,\bar{A}_{kj} \quad \text{(j can be any column and j is not to be summed)}$$

$\det[A]$ is written in terms of the sum of the products of the elements in the ith row (or jth column) and their corresponding cofactors.

Example 1

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\bar{A}_{11} + a_{12}\bar{A}_{12} + a_{13}\bar{A}_{13}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

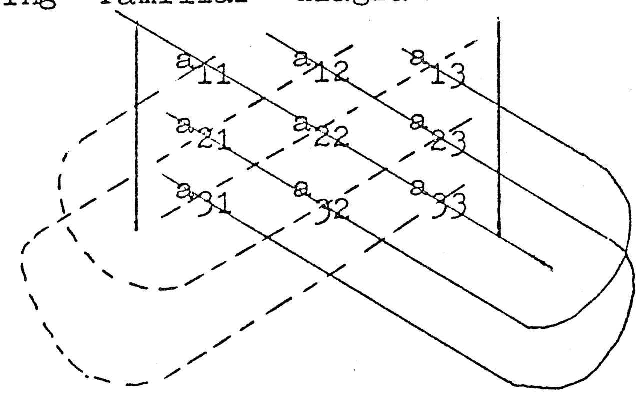
$$= a_{11}a_{22}(-1)^{1+1}a_{33} + a_{11}a_{23}(-1)^{1+2}a_{32}$$

$$-a_{12}a_{21}(-1)^{1+1}a_{33} - a_{12}a_{23}(-1)^{1+2}a_{31}$$

$$+ a_{13}a_{21}(-1)^{1+1}a_{32} + a_{13}a_{22}(-1)^{1+2}a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

In other words, the above operation is in the following "familiar" diagram form:



Example 2   Find $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$

By the diagram method

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 + 0 + 9 - 0 - 0 - 6 = 4$$

Or by the Laplace expansion formula

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\cdot(-1)^2 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$$
$$+ 3(-1)^4 \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$$
$$= 1(-1)^2 \cdot 1 + 0 - 2\cdot 3(-1)^2\cdot 1 + 3\cdot 3(-1)^2\cdot 1$$
$$+ 3\cdot 1(-1)^3\cdot 0$$
$$= 1 - 6 + 9$$
$$= 4$$

2.12.3 Properties of the Determinant

1 The determinant of a matrix and the determinant of the transpose of that matrix are equal. $|A| = |A|^T$

2 $|([A][B])| = |A||B|$

3 Interchanging any two rows or columns changes the sign of the determinant.

4 If all of the elements in a row, or a column, in a matrix are zeros, the determinant is zero.

5 If two rows or two columns in a matrix are identical, the value of the determinant is zero. This is a sufficient condition, but not a necessary condition.

For example, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 0$

6 If matrix [B] is obtained from matrix [A] by adding a multiple of one row of [A] to another (or a multiple of one column to another), then $|A| = |B|$.

Example:

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 4$$

If the second row is replaced by the sum of the second row and twice of the first row, we have

$$|B| = \begin{vmatrix} 1 & 1 & 2 \\ 4 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} = 4 = A$$

2.13 Matrix Inversion (restricted to square matrix)

We are going to cover three methods for matrix inversion.

2.13.1 The Adjoint Method

1 Definition of adjoint matrix:

Adjoint of a matrix $[A]_{mxm}$, written as adj[A] is of the following form,

$$\text{adj}[A] = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{21} & \cdots\cdots & \bar{A}_{m1} \\ \bar{A}_{12} & \bar{A}_{22} & \cdots\cdots & \bar{A}_{m2} \\ \vdots & \vdots & & \vdots \\ \bar{A}_{1m} & \bar{A}_{2m} & \cdots\cdots & \bar{A}_{mm} \end{bmatrix} = \left[\bar{A}_{ij}\right]^T$$

where the adjoint matrix, adj[A], is defined as the transpose of the matrix of cofactors.

Example: If $[A] = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, find adj[A]

First form cofactors: $\bar{A}_{11} = -1$; $\bar{A}_{12} = -5$; $\bar{A}_{13} = 3$;
$\bar{A}_{21} = -3$; $\bar{A}_{22} = 1$; $\bar{A}_{23} = 1$;
$\bar{A}_{31} = 2$; $\bar{A}_{32} = 2$; $\bar{A}_{33} = -6$.

Hence $\text{adj}[A] = \begin{bmatrix} -1 & -3 & 2 \\ -5 & 1 & 2 \\ 3 & 1 & -6 \end{bmatrix}$

2.13.2 Derivation of the Equation for Matrix Inversion

The inverse of a matrix [A], expressed by $[A]^{-1}$, is defined such that

$$[A][A]^{-1} = [I]$$

First let a matrix [P] of order mxm be defined as

$$[P] = [A]\,\text{adj}[A] \quad \text{--------} (1)$$

The term at the ith row and jth column of the matrix [P] is

$$p_{ij} = \sum_{k=1}^{m} a_{ik}\,\bar{A}_{jk} \quad \text{--------} (2)$$

If $i = j$, $p_{ij} = p_{ii} = \sum_{k=1}^{m} a_{ik} \bar{A}_{ik} = \det[A]$

which is precisely the Laplace expansion equation for finding the determinant.

Thus $p_{11} = p_{22} = p_{33} \cdots\cdots\cdots\cdots = p_{mm} = |A|$

If $i \neq j$, $p_{ij} = \sum_{k=1}^{m} a_{ik} \bar{A}_{jk} = 0$

The proof of $p_{ij} = 0$ will be given subsequently.

From the above explanation, it is readily seen that

$$[P] = \begin{bmatrix} |A| & & & \text{All zeros} \\ & |A| & & \\ & & |A| & \\ & & & \ddots \\ \text{All zeros} & & & & |A| \end{bmatrix}$$

$$[P] = |A|\,[I] \text{ ---------- } (3)$$

Substituting equation (3) into equation (1),

$$[A] \operatorname{adj}[A] = |A|\,[I]$$

$$[A] \frac{\operatorname{adj}[A]}{|A|} = [I]$$

Multiplying both sides by $[A]^{-1}$

$$[I] \frac{\operatorname{adj}[A]}{|A|} = [A]^{-1}[I]$$

Finally we arrived at the equation for matrix inversion,

$$\boxed{[A]^{-1} = \frac{adj\,[A]}{det\,[A]}}$$

Example: Inverse the matrix A as defined in Section 12(a)

$$[A]^{-1} = \frac{\operatorname{adj}[A]}{\det[A]} = \begin{bmatrix} -1 & -3 & 2 \\ -5 & 1 & 2 \\ 3 & 1 & -6 \end{bmatrix} \div (-8)$$

$$= \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & -\frac{2}{8} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{2}{8} \\ -\frac{3}{8} & -\frac{1}{8} & \frac{6}{8} \end{bmatrix}$$

$\boxed{\text{Proof that } p_{ij} = 0 \text{ when } i \neq j}$

Let $[A]$ be a 3 by 3 matrix, we can evaluate p_{ij} for both cases that $i = j$ and $i \neq j$.

<u>case i</u> for $i = j = 1$,

$$p_{ij} = p_{11} = \sum_{k=1}^{3} a_{1k}\bar{A}_{1k} = a_{11}\bar{A}_{11} + a_{12}\bar{A}_{12} + a_{13}\bar{A}_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

<u>case ii</u> for $i = 2 \neq j = 1$

$$p_{ij} = p_{21} = \sum_{k=1}^{3} a_{2k}\bar{A}_{1k} = a_{21}\bar{A}_{11} + a_{22}\bar{A}_{12} + a_{23}\bar{A}_{13}$$

$$= a_{21}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \det\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \text{zero}$$

Since the first row and the second row are identical, we conclude that $p_{ij} = 0$ for $i \neq j$.

In general, to evaluate p_{ij} with $i \neq j$ is equivalent to finding the determinant of a matrix with two identical rows of elements a_{ik} with $k = 1$ to m.

2.13.3 Gaussian Elimination Method (Gauss-Jordan Method)

This method can best be explained by a numerical example. Let it be desired to find the inverse of the matrix

$$[A] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

The matrix [A] and an indentity matrix [I] are both subjected to the following elimination procedure simultaneously.

row	matrix [A]			[I]			procedure
1	2	1	1	1	0	0	
2	1	3	1	0	1	0	
3	1	1	4	0	0	1	
4	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}\times$ (first row)
5	0	5/2	1/2	−1/2	1	0	2^{nd} row − 4^{th} row
6	0	1/2	7/2	−1/2	0	1	3^{rd} row − 4^{th} row
7	0	1	1/5	−1/5	2/5	0	$(\frac{2}{5})$ (5th row)
8	0	0	17/5	−2/5	−1/5	1	$-\frac{1}{2}$ (7^{th} row) + 6^{th} row
9	0	0	1	−2/17	−1/17	5/17	$\frac{5}{17}\times 8^{th}$ row
10	1	1/2	0	19/34	1/34	−5/34	$-\frac{1}{2}$ (9^{th} row) + 4^{th} row
11	0	1	0	−3/17	7/17	−1/17	$-\frac{1}{5}$ (9^{th} row) + 7^{th} row
12	1	0	0	11/17	−3/17	−2/17	$-\frac{1}{2}$ (11^{th} row) + 10^{th} row

If we choose 12th, 11th, and 9th column, respectively, we have

1	0	0	11/17	−3/17	−2/17
0	1	0	−3/17	7/17	−1/17
0	0	1	−2/17	−1/17	5/17

The above process is equivalent to that

$$\begin{cases} [A][A]^{-1} = [I] \\ [I][A]^{-1} = [A]^{-1} \end{cases}$$

Therefore the inverse of the matrix [A] is obtained

$$[A]^{-1} = \frac{1}{17}\begin{bmatrix} 11 & -3 & -2 \\ -3 & 7 & -1 \\ -2 & -1 & 5 \end{bmatrix}$$

<u>The use of matrix inverse:</u>

Solve the following simultaneous equations

$$\begin{cases} 2x_1 + x_2 + x_3 = 3 \\ x_1 + 3x_2 + x_3 = -2 \\ x_1 + x_2 + 4x_3 = -6 \end{cases}$$

In matrix form

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -2 \\ -6 \end{Bmatrix}$$

or $[A]\{x\} = \{c\}$

and $\{x\} = [A]^{-1}\{c\}$

Hence

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{1}{17}\begin{bmatrix} 11 & -3 & -2 \\ -3 & 7 & -1 \\ -2 & -1 & 5 \end{bmatrix}\begin{Bmatrix} 3 \\ -2 \\ -6 \end{Bmatrix}$$

Finally

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -1 \\ -2 \end{Bmatrix}$$

2.13.4 Cholesky Scheme

Inverte the matrix [A]

$$[A] = \begin{bmatrix} 4 & 2 & 8 & 6 \\ 3 & 2 & 3 & 2 \\ 6 & 5 & 4 & 3 \\ 2 & 8 & 5 & 3 \end{bmatrix}$$

We have the following three steps:

Step 1 First defining

[T] : an upper triangular matrix with diagonal terms equal to unity

[L] : a lower triangular matrix

Find [L] and [T] such that [A] = [L][T]

[A]				[L]				[T]			
4	2	8	6	l_{11}	0	0	0	1	t_{12}	t_{13}	t_{14}
3	2	3	2	l_{21}	l_{22}	0	0	0	1	t_{23}	t_{24}
6	5	4	3	l_{31}	l_{32}	l_{33}	0	0	0	1	t_{34}
2	8	5	2	l_{41}	l_{42}	l_{43}	l_{44}	0	0	0	1

Evaluate the elements in matrix A in column-wise direction

1^{st} column : $4 = l_{11}$, $3 = l_{21}$, $6 = l_{31}$, $2 = l_{41}$

$$2^{nd}\text{ column}\left\{\begin{array}{l} 2 = l_{11}\, t_{12} , \quad t_{12} = \frac{1}{2} \\ 2 = l_{21}\, t_{12} + l_{22} = \frac{3}{2} + l_{22} , \quad l_{22} = \frac{1}{2} \\ 5 = l_{31}\, t_{12} + l_{32} = 3 + l_{32} , \quad l_{32} = 2 \\ 8 = l_{41}\, t_{12} + l_{42} = 1 + l_{42} , \quad l_{42} = 7 \end{array}\right.$$

$$3^{rd}\text{ column}\left\{\begin{array}{l} 8 = l_{11}\, t_{13} , \quad t_{13} = 2 \\ 3 = l_{21}\, t_{13} + l_{22}\, t_{23} = 3 \times 2 + \frac{1}{2} t_{23} , \quad t_{23} = -6 \\ \text{etc. ....} \end{array}\right.$$

Thus we have

[A]				[L]				[T]			
4	2	8	6	4	0	0	0	1	1/2	2	3/2
3	2	3	2	3	1/2	0	0	0	1	-6	-5
6	5	4	3	6	2	4	0	0	0	1	1
2	8	5	2	2	7	43	-9	0	0	0	1

<u>Step 2</u> Derivation of $[A]^{-1}$

$$[A] = [L][T]$$

$$[A][T]^{-1} = [L][T][T]^{-1} = [L][I] = [L]$$

$$[A]^{-1}[A][T]^{-1} = [A]^{-1}[L]$$

$$[T]^{-1} = [A]^{-1}[L] \text{ ------ } (1)$$

If we can find $[T]^{-1}$, we can find $[A]^{-1}$ from equation (1)

<u>Step 3</u> Find $[T]^{-1}$ through the decomposition procedure that

$$[T][T]^{-1} = [I]$$

[T]				$[T]^{-1}$				[I]			
1	1/2	2	3/2	u_{11}	u_{12}	u_{13}	u_{14}	1	0	0	0
0	1	-6	-5	u_{21}	u_{22}	u_{23}	u_{24}	0	1	0	0
0	0	1	1	u_{31}	u_{32}	u_{33}	u_{34}	0	0	1	0
0	0	0	1	u_{41}	u_{42}	u_{43}	u_{44}	0	0	0	1

We now evaluate the elements in the matrix [T] from the last row backward and row-wise:

last row $\begin{cases} 1 = u_{44} \\ 0 = u_{43} \\ 0 = u_{42} \\ 0 = u_{41} \end{cases}$ 3rd row $\begin{cases} 0 = u_{34} + u_{44}, & u_{34} = -1 \\ 1 = u_{33} + u_{43}, & u_{33} = 1 \\ 0 = u_{32} + u_{42}, & u_{32} = 0 \\ 0 = u_{31} + u_{41}, & u_{31} = 0 \end{cases}$

2nd row $\begin{cases} 0 = u_{24} - 6u_{34} - 5u_{44}, & u_{24} = -1 \\ 0 = u_{23} - 6u_{33} - 5u_{43}, & u_{23} = 6 \\ \text{etc.} \end{cases}$

Thus we have

[T]				$[T]^T$				[I]			
1	1/2	2	3/2	1	$-\frac{1}{2}$	-5	1	1	0	0	0
0	1	-6	-5	0	1	6	-1	0	1	0	0
0	0	1	1	0	0	1	-1	0	0	1	0
0	0	0	1	0	0	0	1	0	0	0	1

It is noted that $[T]^{-1}$, like [T], is also an upper triangular matrix with diagonal terms equal to unity. Such matrix is called "echelon" matrix.

Now we can find $[A]^{-1}$ by the use of equation (1).

$[T]^{-1}$				$[A]^{-1}$				[L]			
1	$-\frac{1}{2}$	-5	1	c_{11}	c_{12}	c_{13}	c_{14}	4	0	0	0
0	1	6	-1	c_{21}	c_{22}	c_{23}	c_{24}	3	1/2	0	0
0	0	1	-1	c_{31}	c_{32}	c_{33}	c_{34}	6	2	4	0.
0	0	0	1	c_{41}	c_{42}	c_{43}	c_{44}	2	7	43	-9

We now evalu te the elements in matrix $[T]^{-1}$ from the last column backward and column-wise,

$$\text{last Column}\begin{cases} 1 = -9\,c_{44}\,, & c_{44} = -1/9 \\ -1 = -9\,c_{34}\,, & c_{34} = 1/9 \\ -1 = -9\,c_{24}\,, & c_{24} = 1/9 \\ 1 = -9\,c_{14}\,, & c_{14} = -1/9 \end{cases}$$

$$3^{rd}\text{ column}\begin{cases} 0 = 4c_{43} + 43\,c_{44} = 4c_{43} + 43/9\,, & c_{43} = \frac{43}{36} \\ 1 = 4c_{33} + 43\,c_{34} = 4c_{33} + 43/9 & c_{33} = -\frac{43}{36} \\ 6 = 4c_{23} + 43\,c_{24} = 4c_{23} + 43/9 & c_{23} = \frac{11}{36} \\ -5 = 4c_{13} + 43\,c_{14} = 4c_{13} - 43/9 & c_{13} = -\frac{1}{18} \end{cases}$$

etc.

Thus we have

$[T]^{-1}$				$[A]^{-1}$				[L]			
1	$-\frac{1}{2}$	-5	1	-14/72	7/9	-1/18	-1/9	4	0	0	0
0	1	6.	-1	5/72	-7/9	11/36	1/9	3	1/2	0	0
0	0	1	-1	-22/72	20/9	-43/36	1/9	6	2	4	0
0	0	0	1	49/72	-29/9	43/36	-1/9	2	7	43	-9

Final Solution

2.15 Eigenvalues and Eigenvectors

If we have an equation in the form that

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (1)$$

or in the symbolic form that

$$[A]\{x\} - \lambda [I]\{x\} = 0 \qquad (1a)$$

it is an eigenvalue problem.

In equation (1), the variables have the following physical meaning:

λ = eigenvalue (natural frequencies or buckling loads)

$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ = eigenvectors (mode shape corresponding to each natural frequency or buckling load)

Solution:

Equation (1) may be written as

$$\begin{bmatrix} (3-\lambda) & 2 & 1 \\ 2 & (2-\lambda) & 1 \\ 0 & 1 & (1-\lambda) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2)$$

or in the symbolic form that

$$[C]\{x\} = \{0\} \qquad (2a)$$

Equation (2) may be solved by the Kramer's rule,

$$x_1 = \frac{\det \begin{bmatrix} 0 & 2 & 1 \\ 0 & (2-\lambda) & 1 \\ 0 & 1 & (1-\lambda) \end{bmatrix}}{\det[C]} = \frac{0}{\det[C]}$$

In order to have a solution for x_1, det[C] must be equal to zero. We thus set det[C] = o which yields the "nontrieval solution".

$$\begin{vmatrix} (3-\lambda) & 2 & 1 \\ 2 & (2-\lambda) & 1 \\ 0 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 6\lambda - 1 = 0$$

$$(\lambda^3 - 1) - (6\lambda^2 - 6\lambda) = 0$$

$$(\lambda - 1)(\lambda^2 + \lambda + 1) - 6\lambda(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 1) = 0$$

We thus find three roots for three eigenvalues:

$$\begin{cases} \lambda_1 = 0.209 \\ \lambda_2 = 1.0 \\ \lambda_3 = 4.791 \end{cases}$$

Corresponding to each eigenvalue λ, we have a set of eigenvectors (x_1, x_2, x_3). We can not find the values for (x_1, x_2, x_3), but if we assume $x_1 = 1$ the other two values for x_2 and x_3 can be found relatively. This is what we call the normalized eigenvector. For example, for $\lambda_1 = 0.209$, we assume $x_1 = 1.0$. Equation (2) becomes

$$\left[\begin{array}{c|cc} (3-\lambda_1) & 2 & 1 \\ \hline 2 & (2-\lambda_1) & 1 \\ 0 & 1 & (1-\lambda_1) \end{array}\right] \begin{Bmatrix} 1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \qquad (3)$$

Since we assumed the first eigenvector x_1, we neglect the first equation in the set (3),

$$\begin{Bmatrix} 2 \\ 0 \end{Bmatrix} + \left[\begin{array}{c|c} 2-\lambda_1 & 1 \\ \hline 1 & 1-\lambda_1 \end{array}\right] \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = 0$$

or

$$\left[\begin{array}{c|c} 1.791 & 1.0 \\ \hline 1.0 & 0.791 \end{array}\right] \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 0 \end{Bmatrix}$$

Following the Kramer's rule

$$x_2 = \frac{\begin{vmatrix} -2.0 & 1.0 \\ 0.0 & 0.791 \end{vmatrix}}{\begin{vmatrix} 1.791 & 1.0 \\ 1.0 & 0.791 \end{vmatrix}} = \frac{-1.582}{0.416} = -3.8$$

$$x_3 = \frac{\begin{vmatrix} 1.791 & -2.0 \\ 1.0 & 0.0 \end{vmatrix}}{\begin{vmatrix} 1.791 & 1.0 \\ 1.0 & 0.791 \end{vmatrix}} = \frac{2}{0.416} = 4.8$$

Hence corresponding to $\lambda_1 = 0.209$, we have a set of normalized eigenvector

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ -3.8 \\ 4.8 \end{Bmatrix} \quad \text{------------} \quad (4)$$

For a vibration problem, λ_1 is the first lowest natural frequence and equation (4) gives the corresponding mode shape. For a buckling problem, λ_1 is the critical buckling load and equation (4) gives the corresponding mode shape.

We can also find the second and the third mode shapes (x_1, x_2, x_3) for the eigenvalues λ_2 and λ_3, respectively.