Real Analysis Bonus Problem

Bold Problem (For Extra Credit)

1. Prove that for every $x \in \mathbb{R}$ there exists a unique $k \in \mathbb{Z}$ such that

   1) $k \leq x$, and
   2) if $m \in \mathbb{Z}$ and $m \leq x$ then $m \leq k$

   (Such k, the greatest integer less than or equal to x, is called the **integer part** of x and is denoted by $\lfloor x \rfloor$)

2. Suppose $\alpha \in \mathbb{R}$ is an irrational number. Prove that the set $S = \{n\alpha - \lfloor n\alpha \rfloor : n \in \mathbb{N}\}$ is dense in $[0,\ 1]$.

1. To prove that such a k is unique, let $x \in \mathbb{R},\ k_1,\ k_2 \in \mathbb{Z}$ such that $k_1 \leq x,\ k_2 \leq x$ and $\forall m \in \mathbb{Z},\ m \leq x \implies m \leq k_1$ and $m \leq k_2$.

   For the sake of contradiction, assume $j \neq k$. Without loss of generality, assume $k_1 < k_2$.

   Consider $m = k_2$.
   Then $m \leq x$ and $m \leq k_2$ but $m > k_2$ which contradicts our assumption.

   $\therefore$ If such a k exists, it is unique.

   To prove such a k exists, let $x \in \mathbb{R},\ M = \mathbb{Z} \cap (-\infty,\ x]$

   $\forall m \in M,\ m \leq x \implies x \in \mathcal{UP}(M) \implies \mathcal{UP}(M) \neq \emptyset$

   Case $1 : 0 \leq x$
   Then $0 \in M$ so $M \neq \emptyset$.

   Case $2 : x < 0$
   Then $0 < -x$.
   By the Archimedean property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $-x < n$

   $-x < n \implies -n < x \implies -n \in M \implies M \neq \emptyset$.

   Then by the corrolary to the well-ordering of the natural numbers, $\max(M) \in M$.

   Let $k = \max(M)$

   Then $k \leq x$ and $\forall m \in \mathbb{Z},\ m \leq x \implies m \leq k$.

   $\therefore \exists! k \in \mathbb{Z}$ such that $k \leq x$ and $\forall m \in \mathbb{Z},\ m \leq x \implies m \leq k$.

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2. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q},\ S = \{n\alpha - \lfloor n\alpha \rfloor : n \in \mathbb{N}\}$.

Claim : $\inf(S) = 0$

By the definition of $\lfloor x \rfloor,\ \forall x \in \mathbb{R},\ \lfloor x \rfloor \leq x$ so $x - \lfloor x \rfloor \geq 0$.
Then $\forall s \in S,\ s \geq 0$ so $0 \in \mathcal{LO}(A)$.

To show that $\forall \gamma > 0,\ \gamma \notin \mathcal{LO}(A)$, let $\gamma > 0$.

$\vdots$

$\therefore (*)\ \inf(S) = 0$

Claim : $\forall s \in S,\ N \in \mathbb{N},\ s < \dfrac{1}{N} \implies \forall n \in \mathbb{N},\ n \leq N,\ ns \in S$

Let $s \in S$ and let $N \in \mathbb{N}$ such that $s < \dfrac{1}{N}$

Then $s = n\alpha - \lfloor n\alpha \rfloor$ for some $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$ such that $m \leq N$.

$$ms = m(n\alpha - \lfloor n\alpha \rfloor) \implies ms = mn\alpha - m\lfloor n\alpha \rfloor$$
$$ms \in S \iff mn\alpha - m\lfloor n\alpha \rfloor \in S \iff m\lfloor n\alpha \rfloor = \lfloor mn\alpha \rfloor$$

$$mn\alpha = m(n\alpha - \lfloor n\alpha \rfloor + \lfloor n\alpha \rfloor) = m(s + \lfloor n\alpha \rfloor) = ms + m\lfloor n\alpha \rfloor$$

$$s < \frac{1}{N} \implies ms < \frac{m}{N}$$
$$m \leq N \implies \frac{m}{N} \leq 1 \implies ms < 1$$

$$m\lfloor n\alpha \rfloor \leq mn\alpha < 1 + m\lfloor n\alpha \rfloor \implies \lfloor mn\alpha \rfloor = m\lfloor n\alpha \rfloor \implies ms \in S$$

$\therefore (**)\ \forall s \in S,\ N \in \mathbb{N},\ s < \dfrac{1}{N} \implies \forall n \in \mathbb{N},\ n \leq N,\ ns \in S$

Let $x,\ y \in [0,\ 1],\ x < y$.
Then $0 < y - x < 1$.

By $(*)$, $\exists s_0 \in S$ such that $0 < s_0 < y - x$.

$0 < s_0 < y - x \implies x < x + s_0 < y$

Let $N = \lfloor \frac{1}{s_0} \rfloor + 1$
Then $N > \frac{1}{s_0} \implies s_0 < \frac{1}{N}$
Then by $(**)$, $\forall n \in \mathbb{N},\ n \le N,\ ns_0 \in S$.

By the Archimedean property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $x < ns_0$.
By the well-ordering of $\mathbb{N}$, we can choose the least such n.

Then $(n-1)s_0 \le x < ns_0$

$(n-1)s_0 \le x < 1 \implies n - 1 < \frac{1}{s_0} \implies n < \frac{1}{s_0} + 1 \implies$
$n \le \lfloor \frac{1}{s_0} \rfloor + 1 \implies n \le N \implies ns_0 \in S$

$(n-1)s_0 \le x < ns_0 \implies (n-1)s_0 \le x < (n-1)s_0 + s_0 \le x + s_0 < y \implies x < ns_0 < y$

$\therefore \forall x,\ y \in [0,\ 1],\ x < y,\ \exists s \in S$ such that $x < s < y$.

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