Chapter 7

Applications of Differential Calculus

Ex 7.1

Question 1.
A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.
(i) Find the average velocity of the points between t = 3 and t = 6 seconds.
(ii) Find the instantaneous velocities at t = 3 and t = 6 seconds.
Solution:
(i) Given $s = 2t^2 + 3t$
(i.e.) $f(t) = 2t^2 + 3t$
Now $f(3) = 18 + 9 = 27 = f(a)$
$f(6) = 72 + 18 = 90 = f(b)$

$$\text{Now } \frac{f(b)-f(a)}{b-a} = \frac{90-27}{6-3} = \frac{63}{3}$$

(ii) $s = 2t^2 + 3t$

$\frac{ds}{dt} = 4t + 3$

at t = 3 Instantaneous velocity

$\frac{ds}{dt} = 4(3) + 3 = 15$ m/s

at t = 6, Instantaneous velocity

$\frac{ds}{dt} = 4(6) + 3 = 27$ m/s

Question 2.
A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.
(i) How long does the camera fall before it hits the ground?
(ii) What is the average velocity with which the camera falls during the last 2 seconds?
(iii) What is the instantaneous velocity of the camera when it hits the ground?
Solution:
(i) $s = 16t^2$
$16t^2 = 400$

$$t^2 = \frac{400}{16} = 25$$

t = 5sec

(ii) Last 2 seconds means $t = 3$ to $t = 5$
$f(t) = 16t2$
$f(3) = 16(9) = 144 = f(a)$
$f(5) = 16(25) = 400 = f(b)$

So $\frac{f(b)-f(a)}{b-a} = \frac{400-144}{5-3} = \frac{256}{2}$

$= 128$ ft/sec

(iii) $s = 16t^2$
$\frac{ds}{dt} = 32\ t$
When it hits the ground at time 5 seconds at $t = 5$,
instantaneous velocity dsdt $= 32\ (5)$
$= 160$ ft/sec.

Question 3.
A particle moves along a line according to the law $s(t) = 2t^3 - 9t2 + 12t - 4$, where $t \geq 0$.
(i) At what times the particle changes direction?
(ii) Find the total distance travelled by the particle in the first 4 seconds.
(iii) Find the particle's acceleration each time the velocity is zero.
Solution:
(i) $s = f(t) = 2t^3 - 9t^2 + 12t - 4$
$V = f'(t) = 6t^2 - 18t + 12$
$V = 0 \Rightarrow 6(t^2 - 3t - 2) = 0$
$(t - 1)(t - 2) = 0$
$t = 1, 2$
When $t < 1$, (say $t = 0.5$)
$V = 6(0.25 - 1.5 + 2) = +ve$
When $1 < t < 2$, (say $t = 1.5$) $V = 6(2.25 - 4.5 + 2) = -ve$ When $t > 2$, (say $t = 3$)
$V = 6(9 - 6 + 2) = +ve$
So the particle changes its direction when t lies between 1 and 2 secs.
(ii) Total distance travelled in first 4 seconds
$s\ (0) = -4$
$s\ (1) = 2(1)^3 - 9(1)^2 + 12(1) - 4$
$= 2 - 9 + 12 - 4$
$= 1$
$s\ (2) = 2\ (2)^3 - 9\ (2)^2 + 12\ (2) - 4$
$= 16 - 36 + 24 - 4$
$= 0$
$s\ (3) = 2(3)^3 - 9(3)^2 + 12\ (3) - 4$
$= 54 - 81 + 36 - 4$
$= 5$
$s\ (4) = 2\ (4)^3 - 9\ (4)^2 + 12\ (4) - 4$
$= 128 - 144 + 48 - 4$
$= 28$

Total distance travelled in the first 4 seconds.
$= |s(0) - s(1)| + |s(1) - s(2)| + |s(2) - s(3)| + |s(3) - s(4)|$
$= |-4 - 1| + |1 - 0| + |0 - 5| + |5 - 28|$
$= |-5| + |1| + |-5| + |-23|$
$= 5 + 1 + 5 + 23$
$= 34$ m

(iii) $$V = \frac{ds}{dt} = f'(t) = 6t^2 - 18t + 12$$

$$V = 0 \Rightarrow 6(t^2 - 3t + 2) = 0$$

$$\Rightarrow (t-1)(t-2) = 0 \Rightarrow t = 1 \text{ or } 2$$

Now, $$a = \frac{dV}{dt} = 12t - 18$$

a (at V = 0) is 'a' at t = 1 and 2
Now a (at t = 1) = 12 – 18 = -6m / sec^2
a (at t = 2) = 24 – 18 = +6m / sec^2

Question 4.
If the volume of a cube of side length x is $V = x^3$. Find the rate of change of the volume with respect to x when x = 5 units.

Solution:

$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$\text{at } x = 5, \frac{dV}{dx} = 3(5^2) = 3(25) = 75 \text{ units}$$

Question 5.
If the mass m(x) (in kilograms) of a thin rod of length x (in metres) is given by, m(x) = 3x−−√ then what is the rate of change of mass with respect to the length when it is x = 3 and x = 21 metres.

Solution:

$$m = \sqrt{3x} = \sqrt{3}\sqrt{x}$$

$$\frac{dm}{dx} = \sqrt{3}\left[\frac{1}{2\sqrt{x}}\right]$$

$$\frac{dm}{dx} (\text{at } x = 3) = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \text{ kg/m}$$

$$\frac{dm}{dx} (\text{at } x = 27) = \frac{\sqrt{3}}{2\sqrt{27}} = \frac{\sqrt{3}}{2(3)\sqrt{3}} = \frac{1}{6} \text{ kg/m}$$

Question 6.
A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?
Solution:

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

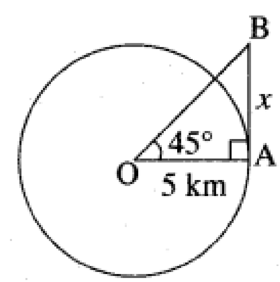
Here $r = 5$ cm and $\frac{dr}{dt} = 2$ cm / sec

$$\therefore \frac{dA}{dt} = \pi(2)(5)(2) = 20\pi \text{ sq.cm / sec}$$

Question 7.
A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?
Solution:
Time for 1 rotation = 10 sec

So angular velocity $= \frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$

From the diagram $\tan 45° = \frac{x}{5} = 1 \Rightarrow x = 5$



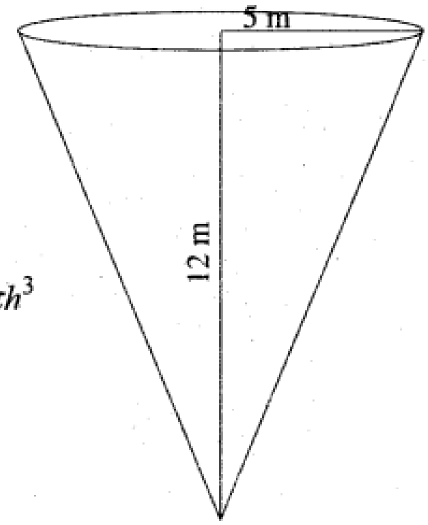
Again $\tan\theta = \frac{x}{5} \Rightarrow x = 5\tan\theta$

$$\Rightarrow \frac{dx}{dt} = 5\sec^2\theta\frac{d\theta}{dt} = 5[\sec^2(45°)]\frac{\pi}{5}$$

$$= 5\left(\sqrt{2}\right)^2\left(\frac{\pi}{5}\right) = \left(\sqrt{2}\right)^2\pi = 2\pi \text{ km/sec}$$

Question 8.
A conical water tank with a vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate of 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?
Solution:

Radius of cone $r = 5$m. Height of cone $h = 12$ m,



$$\therefore \frac{r}{h} = \frac{5}{12} \Rightarrow r = \frac{5}{12}h$$

$$\text{Volume of cone (V)} = \frac{1}{3}\pi r^2 h$$

$$(i.e)\ V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 (h) = \frac{25}{432}\pi h^3$$

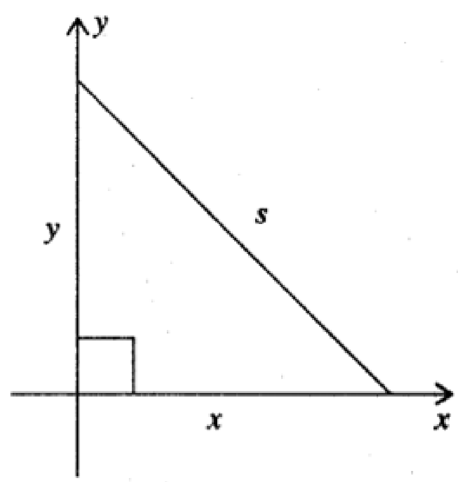
$$\frac{dV}{dt} = \frac{25\pi}{432}\left[3h^2\frac{dh}{dt}\right]$$

Here $h = 8$m and $\frac{dV}{dt} = 10$ cubic m

$$\Rightarrow 10 = \frac{25\pi}{432}(3)(8^2)\frac{dh}{dt} \Rightarrow 10 \times \frac{432}{25\pi} \times \frac{1}{3\times 64} = \frac{dh}{dt}$$

$$(i.e)\ \frac{dh}{dt} = \frac{9}{10\pi} \text{ m/minute}$$

Question 9.
A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the Waif at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall
(i) How fast is the top of the ladder moving down the wall?
(ii) At what rate, the area of the triangle formed by the ladder, wall, and floor, is changing?
Solution:



(i) $x^2 + y^2 = s^2 = 17^2$

When $x = 8$

$$8^2 + y^2 = 17^2$$
$$y^2 = 17^2 - 8^2 = 15^2 \Rightarrow y = 15$$
$$x^2 + y^2 = 17^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$(\div \text{ by } 2)\ x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Here $x = 8$m, $\frac{dx}{dt} = 5$m/sec and $y = 15$m

To find $\frac{dy}{dt}$:

$$(8)(5) + (15)\left(\frac{dy}{dt}\right) = 0$$

$$\therefore \frac{dy}{dt} = \frac{-8\times5}{15} = \frac{-8}{3} \text{ m/sec.}$$

So the top of the ladder is moving down the wall at $\frac{-8}{3}$ m/sec

$$\text{Area} = A = \frac{1}{2} b h$$

$$\therefore \frac{dA}{dt} = \frac{1}{2}\left[b\frac{dh}{dt} + h\frac{db}{dt}\right]$$

Here b = base = 8 m and h = height = 15m

$$\frac{db}{dt} = 5\text{m / sec and } \frac{dh}{dt} = \frac{-8}{3} \text{ m/sec.}$$

$$\therefore \frac{dA}{dt} = \frac{1}{2}\left[(8)\left(\frac{-8}{3}\right) + (15)(5)\right]$$

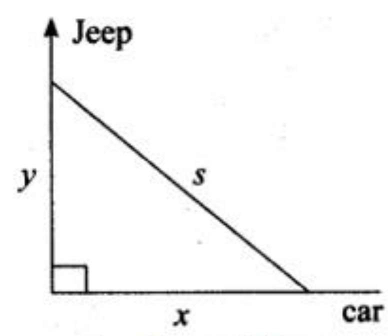
$$= \frac{1}{2}\left[\frac{-64}{3} + 75\right] = \frac{161}{6} = 26.83 \text{ sq m/sec}$$

(ii)

Question 10.
A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Solution:

$$s = \sqrt{x^2 + y^2}$$
$$= \sqrt{0.8^2 + 0.6^2}$$
$$= \sqrt{1} = 1$$



Given $x = 0.8$, $y = 0.6$, $\frac{dy}{dt} = -60$, $\frac{ds}{dt} = 20$

From the figure $s^2 = x^2 + y^2$

differentiate w.r to t

$$
\begin{aligned}
2s\,\frac{ds}{dt} &= 2x\,\frac{dx}{dt} + 2y\frac{dy}{dt} \\
(\div \text{ by } 2) \Rightarrow \quad s\,\frac{ds}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt} \\
1(20) &= (0.8)\,\frac{dx}{dt} + (0.6)\,(-60) \\
\frac{dx}{dt} &= \frac{20+36}{0.8} = \frac{56}{0.8} = 70 \\
\frac{dx}{dt} &= 70 \text{ km/hr}
\end{aligned}
$$

Ex 7.2

Question 1.
Find the slope of the tangent to the curves at the respective given points.
(i) $y = x^4 + 2x^2 - x$ at $x = 1$
(ii) $x = a\cos^3 t, \sin^3 t$ at $t = \frac{\pi}{2}$

Solution:
(i) $y = x^4 + 2x^2 - x$ at $x = 1$

The slope of the tangent at (x, y) is the value of $\frac{dy}{dx}$ at (x_1, y_1)

$$\frac{dy}{dx} = 4x^3 + 4x - 1$$

$$\frac{dy}{dx} \text{ (at } x = 1) = 4 + 4 - 1 = 7$$

(*i.e*) slope of the tangent = 7

(*ii*) $x = a\cos^3 t, y = b\sin^3 t$

$$\frac{dx}{dt} = a[3\cos^2 t](-\sin t)$$

$$\frac{dy}{dt} = b[3\sin^2 t](\cos t)$$

$$\frac{dy}{dx} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t} = \frac{-b}{a}\frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} \left(\text{at } t = \frac{\pi}{2}\right) = \frac{-b}{a}\frac{(1)}{0} = \infty$$

Slope of the tangent $= \infty$

Question 2.
Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$.

Solution:
Let (x_1, y_1) be the required point.

$$y = x^2 - 5x + 4$$
$$\frac{dy}{dx} = 2x - 5$$
$$\frac{dy}{dx} \text{ at } (x_1, y_1) = 2x_1 - 5 = m_1 = \text{slope of the tangent}$$
$$3x + y = 7$$
$$3 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -3 = m_2$$

Given tangent is similarly to the line
$\Rightarrow m_1 = m_2$
$\Rightarrow 2x_1 - 5 = -3$
$\Rightarrow 2x_1 = -3 + 5 = 2$
$\Rightarrow x_1 = 1$
Substituting $x_1 = 1$ in the curve.
$y_1 = 1 - 5 + 4 = 0$
So the required point is (1, 0)

Question 3.
Find the points on the curves $= x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.
Solution:
Let (x_1,y_1) be the required point
$y = x3 - 6x2 + x + 3$
$$\frac{dy}{dx} = 3x^2 - 12x + 1$$
$$\frac{dy}{dx} \text{ at } (x_1, y_1) = 3x_1^2 - 12x_1 + 1 = m = \text{slope of the tangent}$$
$$\therefore \text{ Slope of the normal} = \frac{-1}{m} = \frac{-1}{3x_1^2 - 12x_1 + 1} = m_1 \text{ (say)}$$

To find Slope of the line $x + y = 1729$
$$1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1 = m_2$$

Given that normal is parallel to the line. $m_1 = m_2$
$$(i.e) \frac{-1}{3x_1^2 - 12x_1 + 1} = -1$$
$$3x_1^2 - 12x_1 + 1 = 1 \Rightarrow 3x_1^2 - 12x_1 = 0$$
$$3x_1(x_1 - 4) = 0$$
$$x_1 = 0 \text{ (or) } 4$$

Substituting x_1 values in the curve
when $x_1 = 4$, $y_1 = 3$
when $x_1 = 4$, $y_1 = 4^3 - 6(4)^2 + 4 + 3 = 64 - 96 + 4 + 3 = -25$
So the required points are (0, 3) and (4, -25)

Question 4.
Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.
Solution:
Let the required point be (x_1, y_1) $y^2 - 4xy = x^2 + 5$
Slope of tangent at (x_1, y_1) is $\frac{dy}{dx}$ at (x_1, y_1)
differentiating the curve w.r.to x

$$2y\frac{dy}{dx} - 4\left[x\frac{dy}{dx} + y\right] = 2x$$

$$\frac{dy}{dx}(2y - 4x) = 2x + 4y$$

$$\frac{dy}{dx} = \frac{2x + 4y}{2y - 4x} = \frac{2(x + 2y)}{2(y - 2x)} = \frac{x + 2y}{y - 2x} \quad \ldots (1)$$

Given that the tangent is horizontal (i.e) tangent is parallel to the x-axis
⇒ Equation of tangent will be of the form y = c

$$\Rightarrow \quad \frac{dy}{dx} = 0 \quad \ldots (2)$$

from (1) and (2) we get

$$\frac{x + 2y}{y - 2x} = 0 \Rightarrow x + 2y = 0 \Rightarrow x = -2y$$

Substituting $x = -2y$ in the equation of the curve we get
$y^2 - 4(-2y)(y) = 4y^2 + 5$
$\Rightarrow y^2 + 8y^2 - 4y^2 = 5$
$\Rightarrow 5y^2 = 5 \Rightarrow y^2 = 1$
$\Rightarrow y = \pm 1$
when $y = 1$, $x = -2$ and when $y = -1$, $x = 2$ So the points are (2, -1), (-2, 1)

Question 5.
Find the tangent and normal to the following curves at the given points on the curve.

(i) $y = x^2 - x^4$ **at (1, 0)** **(ii)** $y = x^4 + 2e^x$ **at (0, 2)**

(iii) $y = x \sin x$ **at** $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ **(iv)** $x = \cos t, y = 2\sin^2 t$ **at** $t = \frac{\pi}{3}$

Solution:

(i) $y = x^2 - x^4$

$$\frac{dy}{dx} = 2x - 4x^3$$

$$\frac{dy}{dx} \text{ at } (1, 0) = 2 - 4 = -2 = m = \text{Slope of the tangent}$$

Now m = -2, $(x_1, y_1) = (1, 0)$
So equation of the tangent is
$y - y_1 = m(x - x_1)$
(i. e) $y - 0 = -2(x - 1)$
$y = -2x + 2$
$2x + y = 2$
Slope of tangent = m = -2

$$\therefore \text{Slope of normal} = \frac{-1}{m} = \frac{1}{2}; (x_1, y_1) = (1, 0)$$

So equation of normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$$(i.e)\ y - 0 = \frac{1}{2}(x - 1)$$

$$2y = x - 1$$

$$x - 2y = 1$$

(ii) $y = x^4 + 2e^x$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

$$\frac{dy}{dx} \text{ at } (0, 2) = 0 + 2(1) = 2 = m = \text{slope of tangent}$$

Now $m = 2$; $(x_1, y_1) = (0, 2)$

Equation of tangent is $y - 2 = 2(x - 0)$

$$y - 2 = 2x$$

$$2x - y = -2$$

Slope of normal $= \frac{-1}{m} = \frac{-1}{2}$

Equation of normal is $y - 2 = \frac{-1}{2}(x - 0)$

$$2y - 4 = -x$$

$$x + 2y = 4$$

(*iii*) $y = x \sin x$

$$\frac{dy}{dx} = x \cos x + \sin x$$

$$\frac{dy}{dx} \text{ at } \left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 0 + 1 = 1 = m$$

$$\text{Slope of the tangent } (x_1, y_1) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\text{Equation of tangent is } y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$$

$$x - y = 0$$

$$\text{Slope of normal} = \frac{-1}{m} = \frac{-1}{1} = -1$$

$$\text{Equation of normal is } y - \frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)$$

$$(i.e)\ x + y = \pi$$

(*iv*) $x = \cos t, y = 2 \sin^2 t$

$$x = \cos t;\ \frac{dx}{dt} = -\sin t$$

$$y = 2 \sin^2 t\ ;\ \frac{dy}{dt} = 2\,(2 \sin t \cos t) = 4 \sin t \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{4 \sin t \cos t}{-\sin t} = -4 \cos t$$

$$\frac{dy}{dx}\ \left(\text{at } t = \frac{\pi}{3}\right) = -4\left(\frac{1}{2}\right) = -2 = m = \text{slope of tangent}$$

$$x_1 = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$y_1 = 2\left(\frac{\sqrt{3}}{2}\right)^2 = 2\left(\frac{3}{4}\right) = \frac{3}{2}$$

$$\text{(i.e) } (x_1, y_1) = \left(\frac{1}{2}, \frac{3}{2}\right)$$

Equation of tangent is $y - \frac{3}{2} = -2\left(x - \frac{1}{2}\right)$

$$2y - 3 = -2(2x - 1)$$
$$2y - 3 = -4x + 2 \Rightarrow 4x + 2y = 5$$

Slope of tangent $= m = -2$

$\therefore$ Slope of normal $= \frac{-1}{m} = \frac{1}{2}$

Equation of normal is $y - \frac{3}{2} = \frac{1}{2}\left(x - \frac{1}{2}\right)$

$$2y - 3 = \frac{2x - 1}{2}$$
$$4y - 6 = 2x - 1$$
$$2x - 4y = -5$$

Question 6.
Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.
Solution:

$$y = 1 + x^3$$
$$\frac{dy}{dx} = 3x^2$$

$\frac{dy}{dx}$ at $(x_1, y_1) = 3x_1^2 = m_1 =$ Slope of the tangent.

To find slope of $x + 12y = 12$

$$1 + 12\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-1}{12} = m_2$$

Given that tangent is orthogonal to the lines.

$\Rightarrow$ Product of slopes $= -1$

$$\Rightarrow (3x_1^2)\left(-\frac{1}{12}\right) = -1$$
$$\Rightarrow (3x_1^2) = 12 \Rightarrow x_1^2 = \frac{12}{3} = 4 \Rightarrow x_1 = \pm 2$$

Substituting x_1 values in the curve
when $x_1 = 2, y_1 = 9$; when $x_1 = -2, y_1 = -1$
So the points are (2, 9) and (-2, -7)
To find the equations of tangents:
Tangents are orthogonal to $x + 12y = 12$
So equations of tangents will be of the form $12x - y = k$
The tangent passes through (2, 9) ⇒ $24 - 9 = k \Rightarrow k = 15$.
∴ Equation of tangent is $12x - y = 15$
The tangent passes through (-2, -7) ⇒ $12(-2) + 7 = k \Rightarrow -17$
So equation of tangent is $12x - y = -17$

Question 7.
Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.
Solution:

$$y = \frac{x+1}{x-1}$$

$$\frac{dy}{dx} = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$\frac{dy}{dx} \text{ at } (x_1, y_1) = \frac{-2}{(x_1-1)^2} = m_1 = \text{Slope of tangent}$$

Tangent is parallel the line $x + 2y = 6$
⇒ The slope of tangent = Slope of a line

Slope of $x + 2y = 6$ is $-\frac{1}{2}$

So $\frac{-2}{(x_1-1)^2} = -\frac{1}{2} \Rightarrow (x_1 - 1)^2 = 4$

$$x_1^2 + 1 - 2x_1 - 4 = 0$$

$$\Rightarrow x_1^2 - 2x_1 - 3 = 0 \Rightarrow (x_1 - 3)(x_1 + 1) = 0 \Rightarrow x_1 = 3 \text{ or } -1$$

Substituting x_1 values in the curve $y = \frac{x+1}{x-1}$

when $x_1 = -1, y_1 = 0$; when $x_1 = 3, y_1 = 2$
So the points are (-1, 0) and (3, 2). The tangents are parallel to $x + 2y = 6$.
So the equation of tangents will be of the form $x + 2y = k$.
∴ Equation of tangent is $x + 2y = -1$
The tangent passes through (-1, 0) ⇒ $-1 = k$

The tangent passes through (3, 2) ⇒ 3 + 4 = k ⇒ k = 7
∴ Equation of tangent is x + 2y = 7

Question 8.
Find the equation of tangent and normal to the curve given by x = 7 cos t and y = 2sin t, t ∈ R at any point on the curve.
Solution:

$x = 7\cos t \qquad y = 2\sin t$

$\frac{dx}{dt} = -7\sin t \qquad \frac{dy}{dt} = 2\cos t$

$\frac{dy}{dx}\,[\text{at } 't'] = \frac{2\cos t}{-7\sin t} = m = \text{slope of the tangent}$

$(x_1, y_1) = (7\cos t, 2\sin t)$

Equation of tangent is $y - 2\sin t = \frac{-2\cos t}{7\sin t}(x - 7\cos t)$

$$7y\sin t - 14\sin^2 t = -2x\cos t + 14\cos^2 t$$
$$2x\cos t + 7y\sin t = 14(\cos^2 t + \sin^2 t) = 14$$
$$2x\cos t + 7y\sin t = 14$$

Slope of tangent $= m = \frac{-2\cos t}{7\sin t}$

∴ Slope of normal $= \frac{-1}{m} = \frac{7\sin t}{2\cos t}$

$$(x_1, y_1) = (7\cos t, 2\sin t)$$

So equation of normal is

$$y - 2\sin t = \frac{7\sin t}{2\cos t}(x - 7\cos t)$$

(2 cos t)y – 4 sint cos t = (7 sin t)x – 49 sin t cos t
x(7 sin t) – y(2 cos t) = 45 sin t cos t

Question 9.
Find the angle between the rectangular hyperbola xy = 2 and the parabola $x^2 + 4y = 0$.
Solution:
Solving the given two equations. To find the point of intersection:

$$x^2 + 4y = 0$$
$$4y = -x^2 \Rightarrow y = -\frac{x^2}{4}$$
Substituting $y = -\frac{x^2}{4}$ in $xy = 2$ we get
$$x\left(\frac{-x^2}{4}\right) = 2 \Rightarrow -x^3 = 8 \Rightarrow x = -2$$

$$\text{when } x = -2 \Rightarrow y = \frac{-(4)}{4} = -1$$
$$(x_1, y_1) = (-2, -1)$$
$$xy = 2$$
Differentiating w.r. to x
$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$
$$\frac{dy}{dx} \text{ at } (-2, -1) = -\left(\frac{-1}{-2}\right) = \frac{-1}{2} = m_1$$
$$x^2 + 4y = 0$$
Differentiating w.r.to x
$$2x + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{4} = \frac{-x}{2}$$
$$\frac{dy}{dx} \text{ at } (-2, -1) = -\left(\frac{-2}{2}\right) = 1 = m_2$$
If θ is the angle between the curves then
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$
$$(i.e)\ \tan\theta = \left|\frac{-\frac{1}{2} - 1}{1 + \left(\frac{-1}{2}\right)(+1)}\right| = \left|\frac{-\frac{3}{2}}{\frac{1}{2}}\right| = 3 \Rightarrow \theta = \tan^{-1}(3)$$

Question 10.
Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.
Solution:
Let (x_1, y_1) be the point of intersection of the two curves
I Curve: $x^2 - y^2 = r^2$

Differentiating w.r.to x

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

$$\frac{dy}{dx} \text{ at } (x_1, y_1) = \frac{x_1}{y_1} = m_1$$

II Curve: $xy = c^2$

Differentiating w.r.to x

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{dx} \text{ at } (x_1, y_1) = \frac{-y_1}{x_1} = m_2$$

Now $m_1\, m_2 = \left(\frac{x_1}{y_1}\right)\left(\frac{-y_1}{x_1}\right) = -1 \Rightarrow$ the two curves cut orthogonally.

Ex 7.3

Queston 1.
Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

(i) $f(x)=\left|\frac{1}{x}\right|, x \in [-1,1]$ **(ii)** $f(x)=\tan x, x \in [0,\pi]$ **(iii)** $f(x)=x-2\log x, x \in [2,7]$

Solution:

$$(i)\ f(x)=\left|\frac{1}{x}\right|$$

f(x) is not continuous at x = 0. So Rolle's Theorem is not applicable.

(ii) f(x) = tan x, x ∈ [0, π]
The function tan x is not continuous on [0, π] because it is not defined at π2
The function is not differentiable on (0, π), because it is not continuous at π2
and f(0) = tan (0) = 0
f(π) = tan π = 0
f (0) = f(π) = 0
Since 'tan x' is not continuous in [0, π] and not differentiable in (o, π), Rolle's theorem is not applicable.

(iii) f(x) = x – 2 log x, x ∈ [2, 7]
f(x) is continuous on [2, 7] and
f(x) is differentiable on (2, 7)
f(2) = 2 – 2 log 2, f (7) = 7 – 2 log 7
f(2) ≠ f(7)
Hence, Rolle's theorem is not applicable.

Question 2.
Using Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the following functions:

(i) $f(x)=x^2-x,\ x \in [0,1]$ **(ii)** $f(x)=\frac{x^2-2x}{x+2},\ x \in [-1,6]$

(iii) $f(x)=\sqrt{x}-\frac{x}{2}, x \in [0,9]$

Solution:

Tangent is parallel to x axis. So $\frac{dy}{dx}=0$

(i) $f(x)=x^2-x$

$$f'(x)=2x-1$$
$$f'(x)=0 \Rightarrow 2x-1=0 \Rightarrow x=\frac{1}{2} \in [0,1]$$

(ii) $f(x)=\frac{x^2-2x}{x+2}$

$$f'(x)=\frac{(x+2)(2x-2)-(x^2-2x)(1)}{(x+2)^2}=\frac{x^2+4x-4}{(x+2)^2}$$
$$f'(x)=\frac{x^2+4x-4}{(x+2)^2}$$
$$f'(x)=0 \Rightarrow x^2+4x-4=0$$
$$x=\frac{-4\pm\sqrt{16+16}}{2}=\frac{-4\pm4\sqrt{2}}{2}=-2\pm2\sqrt{2}$$

But $x \in [-1, 6]$

$$\therefore x=-2\pm2\sqrt{2}$$

(iii) $f(x)=\sqrt{x}-\frac{x}{3}$

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{3}$$
$$f'(x)=0 \Rightarrow \frac{1}{2\sqrt{x}}-\frac{1}{3}=0$$
$$\Rightarrow \frac{1}{2\sqrt{x}}=\frac{1}{3} \Rightarrow 2\sqrt{x}=3 \Rightarrow \sqrt{x}=\frac{3}{2} \Rightarrow x=\frac{9}{4}$$
$$x=\frac{9}{4} \in [0,9]$$

Question 3.

Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals :

(i) $f(x)=\frac{x+1}{x}, x\in[-1,2]$ **(ii)** $f(x)=|3x+1|, x\in[-1,3]$

Solution:

$(i)\ f(x)=\frac{x+1}{x}$

The function is not continuous at x = 0. So Lagrange's mean value theorem is not applicable in the given interval.

(ii) f(x) = |3x + 1|, x ∈ [-1, 3]
3x + 1 = 0
x = $-\frac{1}{3}$
f(x) is not differentiable at x = $-\frac{1}{3}$
Hence, Lagrange's mean value theorem is not applicable.

Question 4.
Using the Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval:
(i) f(x) = $x^3 - 3x + 2$, x ∈ [-2, 2]
(ii) f(x) = (x – 2)(x – 7), x ∈ [3, 11]
Solution:
(i) f(x) = $x^3 - 3x + 2$
Here a = -2, b = 2

$$f(a)=f(-2)=0$$
$$f(b)=f(2)=4$$
$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{4-0}{2+2}=\frac{4}{4}=1 \quad \ldots(1)$$
$$f(x)=x^3-3x+2$$
$$f'(x)=3x^2-3$$
$$f'(c)=3c^2-3 \quad \ldots(2)$$

from (1) and (2) $\quad 3c^2-3=1 \Rightarrow 3c^2=4$

$$c^2=\frac{4}{3} \Rightarrow c=\pm\frac{2}{\sqrt{3}} \in[-2,2]$$

$(ii)\ f(x)=(x-2)(x-7)$

$$f(x)=x^2-9x+14$$
$$f'(x)=2x-9$$
$$f'(c)=2c-9 \quad \ldots(1)$$

Here $a=3,\ b=11$

$$f(a) = f(3) = -4$$
$$f(b) = f(11) = 36$$
$$\frac{f(b)-f(a)}{b-a} = \frac{36+4}{11-3} = 5 \qquad \text{... (2)}$$
$$\text{But } f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 2c - 9 = 5$$
$$2c = 5 + 9 = 14$$
$$c = 7 \in [3, 11]$$

Question 5.
Show that the value in the conclusion of the mean value theorem for

(i) $f(x) = \frac{1}{x}$ **on a closed interval of positive numbers [*a*,*b*] is** $\sqrt{ab}$

(ii) $f(x) = Ax^2 + Bx + C$ **on any interval [*a*, *b*] is** $\frac{a+b}{2}$

Solution:

2

$$f(x) = \frac{1}{x}\ x \in [a, b]$$
$$f(a) = \frac{1}{a};\ f(b) = \frac{1}{b}$$
$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a}$$
$$= \frac{a-b}{(b-a)ab} = \frac{-1}{ab} \qquad \text{...(1)}$$

(i)

$$\text{Now } f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$$
$$\text{So, } f'(c) = \frac{-1}{c^2} \qquad \text{... (2)}$$

from (1) and (2) $\frac{-1}{x^2} = \frac{-1}{ab} \Rightarrow x^2 = ab \Rightarrow x = \pm\sqrt{ab}$ but a and b are positive integers

$$\text{So, } x = \sqrt{ab}$$

$$\text{So, } x = \sqrt{ab}$$

(ii)
$$f(x) = Ax^2 + Bx + C,\ x \in [a, b]$$
$$f'(x) = 2Ax + B$$
$$f'(k) = 2Ak + B \qquad \text{... (1)}$$
$$f(x) = Ax^2 + Bx + C$$
$$\text{So, } f(a) = Aa^2 + Ba + C$$
$$\text{and } f(b) = Ab^2 + Bb + C$$
$$\text{So, } \frac{f(b)-f(a)}{b-a} = \frac{Ab^2+Bb+C-(Aa^2+Ba+C)}{b-a}$$
$$= \frac{Ab^2+Bb+C-Aa^2-Ba-C}{b-a}$$
$$= \frac{A(b+a)(b-a)+B(b-a)}{b-a}$$
$$= A(a+b)+B \qquad \text{...(2)}$$
$$\text{But, } f'(k) = \frac{f(b)-f(a)}{b-a}$$
$$(1) = (2)$$
$$\Rightarrow 2Ak + B = A(a+b) + B$$
$$2Ak = A(a+b)$$
$$\Rightarrow k = \frac{a+b}{2}$$

Question 6.
A racecar driver is racing at 20th km. If his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next two hours?
Solution:

By Langrange's Mean Volume theorem, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Here the interval is [0, 2] and f(0) = 20, f(2) = ?
f(b) – f(a) ≤ (b – a)f’c)
here f (a) = 20
⇒ f(b) – 20 ≤ 150(2 – 0)

⇒ f(b) ≤ 320
(i.e) f(2) = 320 km.

Question 7.
Suppose that for a function f(x), f'(x) ≤ 1 for all 1 ≤ x ≤ 4. Show that f(4) – f(1) ≤ 3.
Solution:

$f'(x) \le 1$ for $1 \le x \le 4$

$$\Rightarrow \frac{f(4)-f(1)}{4-1} \le 1$$

$$\left(\therefore f'(c) = \frac{f(b)-f(a)}{b-a}\right) \Rightarrow f(4)-f(1) \le 3$$

Question 8.
Does there exist a differentiable function f(x) such that f(0) = -1, f(2) = 4 and f'(x) ≤ 2 for all x. Justify your answer.
Solution:
f(0) = -1, f(2) = 4, f(x) ≤ 2

We know $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

Here a = 0, b = 2

$$f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{4-(-1)}{2-0}$$

$$= \frac{4+1}{2} = \frac{5}{2} = 2.5 \notin [0, 2]$$

So this is not possible

Question 9.

Show that there lies a point on the curve $f(x) = x(x+3)e^{\frac{\pi}{2}}, -3 \le x \le 0$ where the tangent is drawn is parallel to the x-axis.
Solution:

$$f(x) = x(x+3)\, e^{\frac{\pi}{2}} \quad x \in [-3, 0]$$

$$f(x) = (x^2 + 3x)\, e^{\frac{\pi}{2}}$$

The tangent is parallel to x axis

$$\Rightarrow \frac{dy}{dx} = 0 \ (i.e) \Rightarrow f(x) = 0$$

$$\text{Now } f'(x) = (2x+3)\, e^{\frac{\pi}{2}}$$

$$f'(x) = 0 \Rightarrow 2x + 3 = 0 \Rightarrow x = -\frac{3}{2} \in [-3, 0]$$

⇒ There lies a point in [-3,0], where the tangent is parallel to the x-axis.

Question 10.
Using mean value theorem prove that for, a > 0, b > 0, |e-a – e-b| < |a – b|
Solution:

$$\text{Let } f(x) = e^{-x}$$

$$f'(x) = -e^{-x}$$

$$\text{Now } f'(c) = \frac{f(b) - f(a)}{b - a} = \left|\frac{e^{-b} - e^{-a}}{b - a}\right| \leq \left|-e^{-c}\right| \leq 1 \Rightarrow \frac{\left|e^{-a} - e^{-b}\right|}{|a - b|} \leq 1$$

$$\Rightarrow \left|e^{-a} - e^{-b}\right| < |a - b|$$

Ex 7.4

Question 1.

Write the Maclaurin series expansion of the following functions:

(i) e^x

(ii) sin x

(iii) cos x

(iv) log (1 – x); $-1 \leq x < 1$

(v) $\tan^{-1}(x)$; $-1 \leq x \leq 1$

(vi) $\cos^2 x$

Solution:

(i)

$$f(x) = e^x \quad ; \quad f(0) = e^0 = 1$$
$$f'(x) = e^x \quad ; \quad f'(0) = 1$$
$$f''(x) = e^x \quad ; \quad f''(0) = 1$$
$$f(x) = e^x = 1 + \frac{1 \cdot x}{1!} + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots$$
$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \text{holds for all } x$$

(ii)

$$f(x) = \sin x \quad ; \quad f(0) = 0$$
$$f'(x) = \cos x \quad ; \quad f'(0) = 1$$
$$f''(x) = -\sin x \; ; \quad f''(0) = 0$$
$$f'''(x) = -\cos x ; \quad f'''(0) = -1$$
$$f^4(x) = \sin x \quad ; \quad f^4(0) = 0$$
$$f^5(x) = \cos x \quad ; \quad f^5(0) = 1$$

The Maclaurin expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x^1}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} + f''(0) + \ldots.$$
$$f(x) = \sin x = \frac{x^1}{\lfloor 1}(1) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^5}{\lfloor 5}(1) + \ldots.$$
$$= x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots.$$

(*iii*)

$$\begin{aligned} f(x) &= \cos x \;; & f(0) &= 1 \\ f'(x) &= -\sin x \;; & f'(0) &= 0 \\ f''(x) &= -\cos x \;; & f''(0) &= -1 \\ f^3(x) &= \sin x \;; & f^3(0) &= 0 \\ f^4(x) &= \cos x \;; & f^4(0) &= 1 \end{aligned}$$

The Maclaurin's expansion $f(x) = f(0) + \frac{x^1}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots$

$$\begin{aligned} f(x) &= \cos x = 1 + \frac{x^2}{\lfloor 2}(-1) + \frac{x^4}{\lfloor 4}(1) - \ldots \\ &= 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots \end{aligned}$$

(*iv*)

$$\begin{aligned} f(x) &= \log(1-x) \\ f(0) &= 0 \\ f'(x) &= \frac{1}{1-x}(-1) = \frac{-1}{1-x} \\ f'(0) &= -1 \\ f''(x) &= -\frac{-1}{(1-x)^2}(-1) = \frac{-1}{(1-x)^2} \\ f''(0) &= -1 \\ f'''(x) &= -\left(\frac{-2}{(1-x)^3}(-1)\right) = \frac{-2}{(1-x)^3} \\ f'''(0) &= -2 \end{aligned}$$

The Maclaurin's expansion of $f(x)$ is $f(x) = f(0) + \frac{x^1}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots$.

$$\begin{aligned} \text{Here } f(x) &= \log(1-x) \\ \text{So } \log(1-x) &= 0 + \frac{x}{\lfloor 1}(-1) + \frac{x^2}{\lfloor 2}(-1) + \frac{x^4}{\lfloor 4}(-2) \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots \end{aligned}$$

(v)

$$
\begin{aligned}
f(x) &= \tan^{-1} x \quad ; \; f(0) = 0 \\
f'(x) &= \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \; \ldots . \; ; \; f'(0) = 1 = 1! \\
f''(x) &= -2x + 4x^3 - 6x^5 \ldots . \; ; \; f''(0) = 0 \\
f'''(x) &= -2 + 12x^2 - 30x^4 \ldots . \; ; \; f'''(0) = -2 = -(2!) \\
f^{iv}(x) &= 24x - 120x^3 \ldots . \; ; \; f^{iv}(0) = 0 \\
f^{v}(x) &= 24 - 360x^2 \ldots . \; ; \; f^{v}(0) = 24 = 4! \\
\tan^{-1} x &= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{4!}{5!}x^5 + \ldots . \\
&= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots .
\end{aligned}
$$

holds in $|x| \le 1$.

(vi) $f(x) = \cos^2 x$

$f(0) = 1$

$f'(x) = 2 \cos x \, (- \sin x) = - \sin 2x$

$f'(0) = 0$

$f''(x) = (-\cos 2x)(2)$

$f''(0) = -2$

$f'''(x) = -2[- \sin 2x](2) = 4 \sin 2x$

$f'''(0) = 0$

$f4\ (x) = 4(\cos 2x)(2) = 8 \cos 2x$

$f4\ (0) = 8$

Now the Maclaurin's expansion of $f(x)$ is $f(x) = f(0) + \dfrac{x^1}{\lfloor 1} f'(0) + \dfrac{x^2}{\lfloor 2} f''(0) + \ldots .$

$$(i.e.,) \qquad \cos^2 x = 1 + \frac{x}{\lfloor 1}(0) + \frac{x^2}{\lfloor 2}(-2) + \frac{x^3}{\lfloor 3}(0) + \frac{x^4}{\lfloor 4}(8) \ldots .$$

$$(i.e.,) \qquad \cos^2 x = 1 - \frac{2x^2}{\lfloor 2} + \frac{2^3 x^4}{\lfloor 4} \ldots .$$

Question 2.

Write down the Taylor series expansion, of the function log x about x = 1 upto three non-zero terms for x > 0.

Solution:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) \ldots.$$

$$\text{Here} \quad f(x) = \log x \text{ and } a = 1$$

$$\text{so} \quad f(x) = \log x \Rightarrow f(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$$

$$f'''(x) = -\left(\frac{-2}{x^3}\right) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

$$\text{So} \quad \log x = 0 + (x-1)(1) + \frac{(x-1)^2}{\lfloor 2}(-1) + \frac{(x-1)^3}{\lfloor 3}(2) - \ldots.$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots.$$

Question 3.

Expand sin x in ascending powers x – $\frac{\pi}{4}$ upto three non-zero terms.

Solution:

f (x) = sin x

$$x = \left(x - \frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f^4(x) = \sin x \Rightarrow f^4\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{x - \frac{\pi}{4}}{\lfloor 1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{\lfloor 2}\left(-\frac{1}{\sqrt{2}}\right) - \ldots.$$

$$= \frac{1}{\sqrt{2}} \left\{1+\frac{\left(x-\frac{\pi}{4}\right)}{\underline{|1}}-\frac{\left(x-\frac{\pi}{4}\right)^2}{\underline{|2}}-....\right\}$$

$$= \frac{\sqrt{2}}{2} \left[1+\frac{\left(x-\frac{\pi}{4}\right)}{\underline{|1}}-\frac{\left(x-\frac{\pi}{4}\right)^2}{\underline{|2}}-....\right]$$

Question 4.
Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x - 1$
Solution:
$f(x) = x^2 - 3x + 2 = (x - 1)(x - 2)$
$f(1) = 0$
$f'(x) = 2x - 3$; $f'(1) = -1$
$f''(x) = 2$; $f''(1) = 2$

$$f(x) = f(1) + \frac{x-1}{\underline{|1}} f'(1) + \frac{(x-1)^2}{\underline{|2}} f''(1) \ldots.$$

$$= 0 + \frac{x-1}{\underline{|1}} (-1) + \frac{(x-1)^2}{\underline{|2}} (2) \ldots.$$

$$= -(x-1) + (x-1)^2$$

Ex 7.5

Evaluate the following limits, if necessary use l'Hopital Rule.

Question 1.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

Solution:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1-1}{0} = \frac{0}{0} = \text{indeterminate form}$$

so (Applying l'Hôpital Rule we get)

$$\lim_{x\to 0}\frac{\sin x}{2x} = \frac{0}{0} = \text{indeterminate form}$$

so l'Hôpital Rule we get

$$\lim_{x\to 0} = \frac{\cos x}{2} = \frac{1}{2}$$

Question 2.

$$\lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3}$$

Solution:

Put $x=\frac{1}{y}$ so $x\to\infty \Rightarrow y\to 0$

$$= \lim_{y\to 0}\frac{\frac{2}{y^2}-3}{\frac{1}{y^2}-\frac{5}{y}+3}$$

$$= \lim_{y\to 0}\frac{\frac{1}{y^2}\left(2-3y^2\right)}{\frac{1}{y^2}\left(1-5y+3y^2\right)} = \frac{2}{1} = 2$$

Question 3.

$$\lim_{x\to\infty}\frac{x}{\log x}$$

Solution:

$$\lim_{x\to\infty} \frac{x}{\log x} = \left(\frac{\infty}{\infty}\right) = \text{indeterminate form}$$

$\therefore$ Applying L.H. Rule

$$\underset{x\to\infty}{\text{Lt}} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$$

Question 4.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x}$$

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{1}{\cos x}\times\frac{\cos x}{\sin x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin x}$$

$$= \frac{1}{1} = 1$$

Question 5.

$$\lim_{x\to\infty} e^{-x}\sqrt{x}$$

Solution:

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \frac{\sqrt{x}}{e^x} = \left(\frac{\infty}{\infty}\right) = \text{indeterminate form}$$

Applying L.H. Rule

$$\lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x}$$

as $x \to \infty$, denominator $\to 0$

Question 6.

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution:

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \left(\frac{0}{0}\right) = \text{indeterminate form}$$

Applying L.H. Rule

$$\lim_{x\to 0} \frac{1-\cos x}{x\cos x+\sin x} = \frac{1-1}{0+0} = \frac{0}{0} \text{ indeterminate form}$$

Again applying L.H. Rule

$$\frac{\sin x}{-x\sin x+\cos x+\cos x} = \frac{0}{2} = 0$$

Question 7.

$$\lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right).$$

Solution:

$$\lim_{x\to 1^+} \frac{2}{x^2-1} - \lim_{x\to 1^+} \frac{x}{x-1}$$

Applying L.H. Rule

$$\lim_{x\to 1^+} \frac{0}{2x} - \lim_{x\to 1^+} \frac{1}{1} = 0-1 = -1$$

Question 8.

$$\lim_{x\to 0^+} x^x.$$

Solution:

Let $y = x^x$

Taking log on both sides

$$\log y = \log x^x = x\log x$$

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \frac{-\infty}{\infty}$$

which indeterminate form

Applying L.H. Rule

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{x}\left(\frac{-x^2}{1}\right)$$

$$= \lim_{x\to 0^+} (-x) = 0$$

$\log y = 0 \Rightarrow y = e^0 = 1$

Question 9.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Solution:

$$\text{Let } y = \left(1+\frac{1}{x}\right)^x$$

Taking log on both sides

$$\log y = \log\left(1+\frac{1}{x}\right)^x = x\log\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\log y = \lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

Applying L.H. Rule $\Rightarrow$ $\dfrac{\lim\limits_{x\to\infty}\left(\dfrac{1}{1+\frac{1}{x}}\right)\left(\dfrac{-1}{x^2}\right)}{\left(\dfrac{-1}{x^2}\right)}$

$$\lim_{x\to\infty}\log y = \lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right) = 1$$

(*i.e.*,) $\log y = 1 \Rightarrow y = e^1 = e$

Question 10.

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$$

Solution:

$$\text{Let } y = (\sin x)^{\tan x}$$

Taking log on both sides

$$\log y = \log(\sin x)^{\tan x} = \tan x\log(\sin x)$$

$$= \frac{\log\sin x}{\cot x}$$

$$\lim_{x\to\frac{\pi}{2}}\log y = \lim_{x\to\frac{\pi}{2}}\frac{\log\sin x}{\cot x} = \frac{0}{0} = \text{indeterminate form}$$

Applying L.H. Rule

$$\lim_{x\to\frac{\pi}{2}} \log y = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-\operatorname{cosec} x} = 0$$

$(i.e.,)\log y = 0 \Rightarrow y = e^0 = 1$

Question 11.

$\lim\limits_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$.

Solution:

$\lim\limits_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$.

$$\text{Let } y = (\cos x)^{\frac{1}{x^2}}$$

Taking log on both sides

$$\log y = \log(\cos x)^{\frac{1}{x^2}} = \frac{1}{x^2}\log\cos x$$

$$\log y = \frac{\log(\cos x)}{x^2}$$

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\log(\cos x)}{x^2} = \frac{0}{0} = \text{indeterminate form}$$

So Applying L.H. Rule

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \frac{-\tan x}{2x} = \frac{0}{0} = \text{indeterminate form}$$

Again Applying L.H. Rule

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{-\sec^2 x}{2} = \frac{-1}{2}$$

$$(i.e.,) \quad \log y = -\frac{1}{2}$$

$$\Rightarrow y = e^{-\frac{1}{2}} = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}}$$

Question 12.

If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded

continuously, (that is as n $\to \infty$), show that the amount after t years is A = A0e^{rt}.

Solution:

$$\text{Given A} = \lim_{n\to\infty} A_0 \left(1+\frac{r}{n}\right)^{nt}$$

$$\text{Let } y = A_0 \left(1+\frac{r}{n}\right)^{nt}$$

$$\therefore \log y = \log A_0 + (nt) \log \left(1+\frac{r}{n}\right)$$

$$\lim_{n\to\infty} \log y = \log A_0 + \lim_{n\to\infty} \log \left(\frac{1+\frac{r}{n}}{1/nt}\right)$$

Applying L.H. Rule

$$\log y = \log A_0 + \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{r}{n}\right)}\left(0-\frac{r}{n^2}\right)}{\frac{-1}{tn^2}}$$

$$(i.e.,) \quad \lim_{n\to\infty} \log y = \log A_0 + \lim_{n\to\infty} \left(\frac{1}{1+\frac{r}{n}}\right)\left(\frac{-r}{n^2}\right)(-tn^2)$$

$$(i.e.,) \quad \lim_{n\to\infty} \log y = \log A_0 + rt \lim_{n\to\infty} \left(\frac{1}{1+\frac{r}{n}}\right)$$

$$(i.e.,) \quad \lim_{n\to\infty} \log y = \log A_0 + rt\,(1)$$

By composite function theorem

$$\log \left(\lim_{n\to\infty} y\right) = \log A_0 + rt$$

$$= \lim_{n\to\infty} A_0 \left(1+\frac{r}{n}\right)^{nt} = e^{\log A0 + rt}$$

$$= e^{\log A0} \,.\, e^{rt}$$

$$(i.e.,) \quad A = A_0 e^{rt}$$

Ex 7.6

Question 1.

Find the absolute extrema of the following functions on the given closed interval.

(i) f(x) = $x^3 - 12x + 10$; [1, 2]

(ii) f(x) = $3x^4 - 4x3$; [-1, 2]

(iii) $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$; [–1, 1] **(iv)** $f(x) = 2\cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$

Solution:

(i)

$$f(x) = y = x^3 - 12x + 10$$

$$\frac{dy}{dx} = 3x^2 - 12$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

$$\text{Here } x = 2 \in [1, 2]$$

$$\text{Now } f(1) = 1 - 12 + 10 = -1$$

$$f(2) = 8 - 24 + 10 = -6$$

∴ Absolute maximum is – 1 and absolute minimum is – 6

(ii) f(x) = $3x^4 - 4x^3$; [-1, 2]

f'(x) = $12x^3 - 12x^2$

f'(x) = 0 ⇒ $12x^2(x - 1) = 0$

x = 0, 1 ∈ (-1, 2)

Critical points x = 0, 1 and end points of the interval x = -1, 2

f(-1) = 3 + 4 = 7

f(0) = 0 – 0 = 0

f(1) = 3 – 4 = -1

f(2) = 48 – 32 = 16

Absolute maximum f(2) = 16

Absolute minimum f(l) = -1

(iii)

$$f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$$

$$f'(x) = 6 \times \frac{4}{3} x^{\frac{1}{3}} - 3\frac{1}{3} x^{\frac{-2}{3}}$$

$$= 8x^{\frac{1}{3}} - x^{\frac{-2}{3}}$$

$$f'(x) = 0 \Rightarrow 8x^{\frac{1}{3}} - x^{\frac{-2}{3}} = 0$$

$$\Rightarrow 8x^{\frac{1}{3}} = x^{\frac{-2}{3}} = \frac{1}{x^{\frac{2}{3}}}$$

$$\Rightarrow x^{\frac{1}{3}} \cdot x^{\frac{2}{3}} = \frac{1}{8} \quad (i.e.,)\ x = \frac{1}{8}$$

Now $f(-1) = 9$

$f(1) = 3$

and $f\left(\frac{1}{8}\right) = \frac{-9}{8}$

so absolute maximum $= 9$ and absolute minimum $= \frac{-9}{8}$

(iv) f(x) = 2 cos x + sin 2x

f'(x) = -2 sin x + 2 cos 2x

f'(x) = 0 ⇒ cos 2x = sin x

$(i.e.,)\ 1 - 2\sin^2 x = \sin x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}$ or $\sin x = -1$

$\Rightarrow \quad x = \frac{\pi}{6}$ or π

but $x \in \left[0, \frac{\pi}{2}\right]$

So $x = \frac{\pi}{6}$

Now $f(0) = 2$

$f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$

and $f\left(\frac{\pi}{2}\right) = 0$

so absolute maximum $= \frac{3\sqrt{3}}{2}$ and absolute minimum $= 0$

Question 2.

Find the intervals of monotonicities and hence find the local extremum for the following functions

(i) $f(x) = 2x^3 + 3x^2 - 12x$

(ii) $f(x) = \dfrac{x}{x-5}$

(iii) $f(x) = \dfrac{e^x}{1-e^x}$

(iv) $f(x) = \dfrac{x^3}{3} - \log x$

(v) $f(x) = \sin x \cos x + 5, x \in (0, 2\pi)$

Solution:

(i) $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12$

$f'(x) = 0 \Rightarrow 6(x^2 + x - 2) = 0$

(i.e.,) $6(x + 2)(x - 1) = 0$

$\Rightarrow x = -2$ or 1

Taking the points in the number line

The intervals are $(-\infty, -2), (-2, 1), (1, \infty)$

when $x \in (-\infty, -2)$, $f'(x) = 6(-1)(-4) = +$ve

say $x = -3$

$\Rightarrow f(x)$ is strictly increasing in the interval $(-\infty, -2)$ when $x \in (-2, 1)$, $f'(x) = 6(2)(-1) = -$ve

say $x = 0$

$\Rightarrow f(x)$ is strictly decreasing in the interval $(-2, 1)$

when $x \in (1, \infty)$, $f'(x) = 6(4)(+1) = +$ ve say $x = 2$

$\Rightarrow f(x)$ is strictly increasing in $(1, \infty)$

Since f(x) changes from +ve to – ve when passing through -2, the first derivative, test tells us there is a local maximum at $x = -2$ and the local maximum value is $f(-2) = 20$.

Again $f'(x)$ changes from – ve to +ve when passing through $1 \Rightarrow$ there is a local minimum at $x = 1$ and the local minimum value is $f(1) = -7$. So (1) f(x) is strictly increasing on $(-\infty, -2)$ and $(1, \infty)$. And (2) f(x) is strictly decreasing on $(-2, 1)$

The local maximum = 20 and the local minimum = -7

(ii)
$$f(x) = \frac{x}{x-5}$$
$$f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2} = \frac{x-5-x}{(x-5)^2}$$
$$= \frac{-5}{(x-5)^2} < 0$$
(when $x \neq 5$).

f(x) is strictly decreasing on $(-\infty, 5)$ and $(5, \infty)$

And there is no local extremum

(iii)

$$f(x) = \frac{e^x}{1-e^x}$$

$$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2} > 0$$

$$= \frac{e^x}{(1-e^x)^2} > 0$$

For all x values, so f(x) is strictly increasing in (-∞, ∞) and there is no local extremum.

(iv)

$$f(x) = \frac{x^3}{3} - \log x$$

$$f'(x) = \frac{3x^2}{3} - \frac{1}{x} = x^2 - \frac{1}{x}$$

$$f'(x) = 0 \Rightarrow x^2 = \frac{1}{x} \Rightarrow x^3 = 1 \Rightarrow x = 1$$

So the intervals are (0, 1) and (1, ∞)

when $x \in (0, 1)$, $f'(x) = \frac{1}{4} - \frac{1}{2} = -\text{ve}$

say $x = \frac{1}{2}$

⇒ f(x) is strictly decreasing in (0, 1)

when $x \in (1, \infty)$, $f'(x) = 4 - \frac{1}{2} = +\text{ve}$

say x = 2

⇒ f(x) is strictly increasing in (1, ∞)

since f'(x) changes from -ve to +ve at x = 1, there is a local minimum at x = 1 and the local minimum values is $f(1) = \frac{1}{3} - 0 = \frac{1}{3}$

So the function is strictly decreasing on (0, 1) and strictly increasing on (1, ∞) and the local minimum value is $\frac{1}{3}$

(v)

$$\text{Now } f(x) = \sin x \cos x + 5$$

$$= \frac{1}{2}(\sin 2x) + 5$$

$$\Rightarrow \quad f'(x) = \frac{1}{2}(\cos 2x)\, 2 = \cos 2x$$

$$f'(x) = 0 \Rightarrow \cos 2x = 0$$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots .$$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$(\because \ x \in (0, 2\pi))$

So the intervals are $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$

when $x \in \left(0, \frac{\pi}{4}\right)$, $f'(x) = \cos\frac{\pi}{3} = \frac{1}{2} = +\text{ve}$

say $x = \frac{\pi}{6}$

$\Rightarrow f(x)$ is increasing in $\left(0, \frac{\pi}{4}\right)$

when $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

say $x = \frac{\pi}{3}$

$f'(x) = \cos\frac{2\pi}{3} = -\text{ve}$

$\Rightarrow \quad f'(x)$ is strictly decreasing in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

similarly, $f(x)$ is strictly increasing in $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

$f(x)$ is strictly decreasing in $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

and $f(x)$ is strictly increasing in $\left(\frac{7\pi}{4}, 2\pi\right)$

Thus $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$ and strictly decreasing in $\left(\frac{\pi}{4} - \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

So there is a local maximum at $x = \frac{\pi}{4}, \frac{5\pi}{4}$ and the local maximum value is $\frac{11}{2}$. There is a local minimum at $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ and the local minimum value is $\frac{9}{2}$.

Ex 7.7

Question 1.

Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x - 4)^3$

(ii) $f(x) = \sin x + \cos x, 0 < x < 2\pi$

(iii) $f(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$

Solution:

(i) $f(x) = x(x - 4)^3$

$f(x) = x[x^3 - 12x^2 + 48x - 64]$

(i.e.,) $f(x) = x^4 - 12x^3 + 48x^2 - 64x$

$f'(x) = 4x^3 - 36x^2 + 96x - 64$

$f''(x) = 12x^2 - 72x + 96$

$f''(x) = 0 \Rightarrow 12(x^2 - 6x + 8) = 0$

$\Rightarrow 12(x - 2)(x - 4) = 0 \Rightarrow x = 2$ or 4

Marking the points in the number line

The intervals are $(-\infty, 2), (2, 4), (4, \infty)$ when $x \in (-\infty, 2), f''(x) = 12(-2)(-4)$

$[\text{say } x = 0] = +\text{ve}$

in the interval $(-\infty, 2)$ the curve concave upwards

when $x \in (2, 4)$, $f''(x) = (3 - 2)(3 - 4)$

$[\text{say } x = 3] = -\text{ve}$

$\Rightarrow$ The curve concave upwards in $(2,4)$

when $x \in (4, \infty)$, $f''(x) = (5 - 2)(5 - 4)$

$[\text{say } x = 5] = (+)(+) = +\text{ve} \Rightarrow$ The curve concave downwards is $(4, \infty)$ in $(-\infty, 2)$ concave upwards and in $(2, 4)$ the concave downwards

$\Rightarrow x = 2$ is a point of inflection at $x = 2$, $f(2) = -16$

So $(2, -16)$ is a point of inflection

Again in $(2, 4)$ the curve concave downwards and in $(4, \infty)$ the curve concave upwards $\Rightarrow x = 4$ is a point of inflection $f(4) = 0$

$\therefore (4, 0)$ is a point of inflection

So points of inflection are $(2, -16)$ and $(4, 0)$

(ii) f(x) = sin x + cos x

f'(x) = cos x – sin x

f''(x) = – sin x – cos x

f''(x) = 0 ⇒ – sin x = cos x

$$\Rightarrow \frac{\sin x}{\cos x} = -1$$

$$(i.e.,) \quad \tan x = -1$$

$$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

So the intervals are $\left(0, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

when $x \in \left(0, \frac{3\pi}{4}\right)$

say $x = \frac{\pi}{2}$

$$f''(x) = -1 - 0 = -1 = -\text{ve}$$

$\Rightarrow$ in $\left(0, \frac{3\pi}{4}\right)$ the curve concave downwards ....(1)

in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ $f''(x) = 0 - (-1) = 1$

say $x = \pi$ $= +$ ve

$\Rightarrow$ in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ the curve concave upwards ....(2)

similarly $\left(\frac{7\pi}{4}, 2\pi\right)$ the curve concave downwards ....(3)

from (1) and (2) $x = \frac{3\pi}{4}$ is a point of inflection

and from (2) and (3) $x = \frac{7\pi}{4}$ is a point of inflection

Now $f\left(\frac{3\pi}{4}\right) = 0$ and $f\left(\frac{7\pi}{4}\right) = 0$

So the curve concave upwards in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ and concave downwards in $\left(0, \frac{3\pi}{4}\right)$, $\left(\frac{7\pi}{4}, 2\pi\right)$ and the points of inflection are $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

(iii)

$$f(x) = \frac{1}{2}(e^{x} - e^{-x})$$
$$f'(x) = \frac{1}{2}(e^{x} + e^{-x})$$
$$f''(x) = \frac{1}{2}(e^{x} - e^{-x})$$
$$f''(x) = 0 \quad \Rightarrow e^{x} = e^{-x}$$
$$\Rightarrow x = 0$$

So the intervals are $(-\infty, 0)$, $(0, \infty)$ when $x \in (-\infty, 0)$ $f''(x)$ is negative
$\Rightarrow$ the curve concave downwards and when $x \in (0, \infty)$ $f''(x)$ is positive
$\Rightarrow$ the curve concave upwards
$\Rightarrow$ $x = 0$ is a point of inflection $f(0) = \frac{1}{2}(1 - 1) = 0$
So (0, 0) is the point of inflection and the curve concave upwards in $(0, \infty)$ and curve concave downwards in $(-\infty, 0)$ and (0, 0) is the point of inflection.

Question 2.
Find the local extrema for the following functions using second derivative test:
(i) $f(x) = -3x^5 + 5x^3$
(ii) $f(x) = x \log x$
(iii) $f(x) = x^2 e^{-2x}$
Solution:
(i) $f(x) - 3x^5 + 5x^3$
$f'(x) = -15x^4 + 15x^2$
For maximum or minimum $f'(x) = 0$
$\Rightarrow -15x^2(x^2 - 1) = 0$
$x = 0, 1, -1$ [$x = 0$ is not possible as it gives no extremum.
$f'(x) = -60x^3 + 30x$
$f''(-1) > 0 \Rightarrow f(x)$ attains minimum
$f''(1) < 0 \Rightarrow f(x)$ attains maximum
$\therefore$ Local minimum $f(-1) = -3(-1) + 5(-1) = -2$
Local maximum $f(1) = -3(1) + 5(1) = 2$

(ii) $f(x) = x \log x$

$$f'(x) = x\left(\frac{1}{x}\right) + \log x = 1 + \log x$$
$$f''(x) = \frac{1}{x}$$
$$f'(x) = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1$$
$$\Rightarrow x = e^{-1} = \frac{1}{e}$$

$$\text{at } x = \frac{1}{e}, \quad f''(x) = \frac{1}{\frac{1}{e}}^{e} = e > 0 \text{ (+ve)}$$

$$\text{So at } x = \frac{1}{e}, f'(x) = 0 \text{ and } f''(x) = +\text{ve}$$

$$\Rightarrow x = \frac{1}{e} \text{ is a local minimum point and } f\left(\frac{1}{e}\right) = \frac{1}{e}(-1) = \frac{-1}{e}$$

$$\text{So local minimum point is } \left(\frac{1}{e}, \frac{-1}{e}\right) \text{ and local minimum value is } \frac{-1}{e}$$

(iii) $f(x) = x^2$ e-2x

$f'(x) = -2x^2 e^{-2x} + 2x e^{-2x}$

2x For maximum or minimum, $f'(x) = 0$

$\Rightarrow -2x e^{-2x}(x - 1) = 0$

$x = 0, 1$

$f''(x) = -2 [- 2x^2 e^{-2x} + 4x e^{-2x} - e^{-2x}]$

at $x = 0$, $f''(x) > 0 \Rightarrow f(x)$ attains minimum

at $x = 1$, $f''(x) < 0 \Rightarrow f(x)$ attains maximum

∴ Local minimum $f(0) = 0$

Local maximum $f(1) = 1\frac{1}{e2}$

Question 3.

For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

Solution:

$4x^3 + 3x^2 - 6x + 1$

$f'(x) = 12x^2 + 6x - 6$

$f''(x) = 24x + 6$

$f'(x) = 0 \Rightarrow 6(2x^2 + x - 1) = 0$

$6(x + 1)(2x - 1) = 0$

$$x = -1 \text{ or } \frac{1}{2}$$

So the intervals are $(-\infty, -1), (-1, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$

when $x \in (-\infty, -1)$, $f'(x) = 6(-1)(-5)$

say $x = -2 = +\text{ve}$

The function is strictly increasing in $(-\infty, -1)$ ....(1)

when $x \in (-1, \frac{1}{2})$ $f'(x) = (+1), (-1)$

say $x = 0 = -ve$

The function is strictly decreasing in $(-1, \frac{1}{2})$ ....(2)

From (1) and (2)

x = -1 is a maximum point,

and f(-1)= -4 + 3 + 6 + 1 = 6

So local maximum is 6

Again when $x \in (\frac{1}{2}, \infty)$ $f'(x) = (2)(1)$

say $x = 1 = +ve$

$\Rightarrow$ the function is strictly increasing in $(\frac{1}{2}, \infty)$ .... (3)

From (2) and (3) $x = \frac{1}{2}$ is a minimum point and $f(\frac{1}{2}) = \frac{-3}{4}$

So local minimum is $\frac{-3}{4}$

The function is strictly increasing in $(-\infty, -1)$ and $(\frac{1}{2}, \infty)$ and strictly decreasing in $(-1, \frac{1}{2})$. The local maximum is 6 and local minimum is $\frac{-3}{4}$

$$\text{Now } f''(x) = 24x + 6$$

$$f''(x) = 0 \Rightarrow 24x + 6 = 0$$

$$24x = -6$$

$$\Rightarrow x = \frac{-1}{4}$$

So the intervals are $(-\infty, \frac{-1}{4})$ and $(\frac{-1}{4}, \infty)$

when $x \in (-\infty, \frac{-1}{4})$ $f''(x) = -24 + 6$

say $x = -1 = -ve$

$\Rightarrow$ The curve concave downwards in $(-\infty, \frac{-1}{4})$ ....(1)

when $x \in (\frac{-1}{4}, \infty)$ $f''(x) = 6$

say $x = 0 = (+ve)$

$\Rightarrow$ The curve concave upwards in $(\frac{-1}{4}, \infty)$ ....(2)

From (1) and (2) we see that $x = \frac{-1}{4}$ is a point of inflection and $f(\frac{-1}{4}) = \frac{21}{8}$.

So the curve concave upwards in $(\frac{-1}{4}, \infty)$ and concave downwards in $(-\infty, \frac{-1}{4})$ and the point of inflection is $\left(\frac{-1}{4}, \frac{21}{8}\right)$

Ex 7.8

Question 1.
Find two positive numbers whose sum is 12 and their product is maximum.
Solution:
Let the two positive numbers be 'x' and 'y'
Given sum is 12 ⇒ x + y = 12
y = 12 – x
Product P = xy
P = x(12 – x)
P = 12x – x^2
$\frac{DP}{DX}$ = 12 – 2x
For maximum or minimum,
$\frac{DP}{DX}$ = 0 ⇒ 12 – 2x = 0
x = 6
$d^2P/dx^2 = -2$
at x = 6, $d^2P/dx^2 = -2 < 0$
∴ Product 'P' is maximum when x = 6
∴ y = 12 – 6 = 6
Hence, the positive numbers are 6 and 6 and their product is 36.

Question 2.
Find two positive numbers whose product is 20 and their sum is minimum.
Solution:

Let the two numbers be x, $\frac{20}{x}$
Their sum = $x + \frac{20}{x}$
To get the minimum sum

$$s = x + \frac{20}{x}$$

$$s'(x) = 1 - \frac{20}{x^2}$$

$$s''(x) = -20\left(\frac{-2}{x^3}\right) = \frac{40}{x^3}$$

$$s'(x) = 0 \Rightarrow 1 = \frac{20}{x^2} \Rightarrow x^2 = 20$$

$$x = \sqrt{20} = \pm 2\sqrt{5}$$

but x is positive (given)

$$\Rightarrow x = +\,2\sqrt{5}$$

$$\text{when } x = 2\sqrt{5},\ s''(x) = \frac{40}{(2\sqrt{5})^3} = +ve$$

$$\Rightarrow x = +\,2\sqrt{5} \text{ is a minimum point}$$

$$\text{when } x = 2\sqrt{5},\ y = \frac{20}{2\sqrt{5}} = \frac{20}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= 2\sqrt{5}$$

So the two numbers are $2\sqrt{5}$, $2\sqrt{5}$

Question 3.

Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.

Solution:

Given $x + y = 10 \Rightarrow 7 = 10 - x$

To find the smallest value of $x^2 + y^2$

$$\begin{aligned} f(x) &= x^2 + y^2 = x^2 + (10 - x)^2 \\ f'(x) &= 2x + 2(10 - x)(-1) \\ &= 2x - 20 + 2x = 4x - 20 \\ f''(x) &= 4 \\ f'(x) &= 0 \Rightarrow 4x - 20 = 0 \\ &\quad \Rightarrow 4x = 20 \Rightarrow x = 5 \end{aligned}$$

at $x = 5$, $f''(x) = 4$ at $x = 5$, $y = 10 - 5 = 5 = +ve$

$x = 5$ is a minimum point.

So the minimum value of $x^2 + y^2 = 5^2 + 5^2 = 50$

Question 4.

A garden is to be laid out in a rectangular area and protected by a wire fence. What is the largest possible area of the fenced garden with 40 meters of wire?

Solution:

Let the length of the garden be ‘x’ m

Let the breadth of the garden be ‘y’ m

The Garden is fenced with 40 m wire

i.e., Perimeter = $2(x + y) = 40$

$x + y = 20$

$y = 20 - x$

Area of the Garden $A = xy$

$A = x(20 - x)$
$A = 20x - x^2$
$dA/dx = 20 - 2x$
For maximum or minimum,
$dA/dx = 0 \Rightarrow 20 - 2x = 0$
$x = 10$
$d^2A/dx^2 = -2$
at $x = 10$, $d^2A/dx^2 < 0$ Now, $y = 20 - 10 = 10$
Maximum area $= 10 \times 10 = 100$ sq. m;

Question 5.
A rectangular page is to contain 24 cm2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.
Solution:
Let the length of the printed page be = x cm
and breadth = y cm
Now $xy = 24$
$\Rightarrow y = \frac{24}{x}$
The length of the paper $= y + 3$
Area $A = (x + 2)(y + 3)$
$= xy + 3x + 2y + 6$
$= 24 + 3x + 2y + 6$
$= 3x + 2y + 30$ .... (2)
Substituting (1) in (2) we get

$$A = 3x + 2\left(\frac{24}{x}\right) + 30$$

$$A'(x) = 3 + 48\left(\frac{-1}{x^2}\right) = 3 - \frac{48}{x^2}$$

$$A''(x) = -48\left(\frac{-2}{x^3}\right) = \frac{96}{x^3}$$

$$A'(x) = 0 \Rightarrow 3x^2 = 48$$

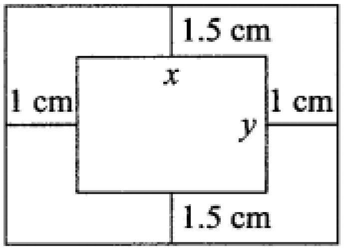
$$x^2 = 16 \Rightarrow x = 4$$

$$at\ x = 4,\ A''(x) = \frac{96}{4^3} = +\text{ ve is a minimum point}$$

$$\text{So at } x = 4,\ y = \frac{24}{x} = \frac{24}{4} = 6$$

∴ Dimensions of the paper are
$x + 2 = 4 + 2 = 6$ cm

and y + 3 = 6 + 3 = 9 cm



Question 6.
A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq. mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?
Solution:
Given Area = 180000 sq. meters
Let length be = x
and breadth be = y

$$\Rightarrow y = \frac{180000}{x} \quad ....(1)$$

$$\text{Now perimeter} = 2x + y \quad (\because \text{one side is along the river})$$

$$\text{Now } p = 2x + \frac{180000}{x} \text{ from (1)}$$

$$\frac{dp}{dx} = 2 - \frac{180000}{x^2}$$

$$\frac{d^2p}{dx^2} = -180000\left(\frac{-2}{x^3}\right) = \frac{360000}{x^3} = +\text{ve}$$

$$\frac{dp}{dx} = 0 \Rightarrow 2 = \frac{180000}{x^2}$$

$$x^2 = 90000$$

$$x = \sqrt{90000} = 300 \text{ m}$$

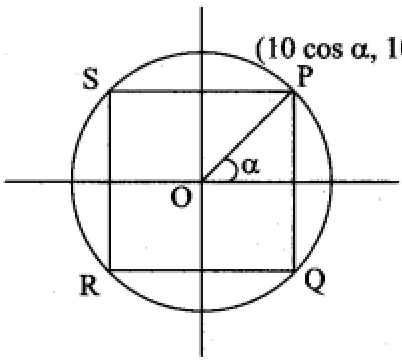
at $x = 300$ m, p'' is positive $\Rightarrow x = 300$ is a minimum point

$$\text{when } x = 300 \text{ m}, \ y = \frac{180000}{300} = 600 \text{ m}$$

$$\text{So minimum perimeter} = 2x + y$$

$$= 2(300) + 600 = 1200 \text{ m}$$

Question 7.
Find the dimensions of the rectangle with the maximum area that can be inscribed in a circle of a radius of 10 cm.
Solution:
P is a point on the circumference of a circle of radius 10 cm P = (10 cos α, 10 sin α)
∴ PQ = 20 sinα and
PS = 20 cos α
A = area of PQRS = (20 sin α) (20 cos α)
= 400 sin α cos α
= (200) (2 sin α cos α)



$$= 200 \sin 2\alpha$$

$$\frac{dA}{d\alpha} = (200)(\cos 2\alpha)(2) = 400 \cos 2\alpha$$

$$\frac{d^2A}{d\alpha^2} = 400(-\sin 2\alpha)\, 2 = -800 \sin 2\alpha$$

$$\frac{dA}{d\alpha} = 0 \Rightarrow \cos 2\alpha = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$$

$$\text{at } \alpha = \frac{\pi}{4} \Rightarrow \frac{d^2A}{d\alpha^2} = (-800)(1) = -\text{ve}$$

$\Rightarrow$ Area is maximum when $\alpha = \frac{\pi}{4}$

So $PQ = 20\left(\frac{1}{\sqrt{2}}\right) = \frac{20 \times \sqrt{2}}{\sqrt{2}\sqrt{2}} = 10\sqrt{2}$

and $PS = 20\left(\frac{1}{\sqrt{2}}\right) = 10\sqrt{2}$

So the dimensions of the rectangle are $10\sqrt{2}$ cm, $10\sqrt{2}$ cm,

Question 8.
Prove that among all the rectangles of the given perimeter, the square has the maximum

area.

Solution:

Let the length and breadth of the rectangle be x and y respectively.

P = 2(x + y) [given]

$$\Rightarrow \quad x+y \quad = \quad \frac{P}{2} \text{ (or) } y = \frac{P}{2} - x$$

$$\text{Area} \quad = \quad A = xy = x\left[\frac{P}{2} - x\right]$$

Substitute (3) in (2) we get

$$V = x^2\left(\frac{108-x^2}{4x}\right) = 27x - \frac{x^3}{4}$$

$$V = 27x - \frac{x^3}{4}$$

$$\frac{dV}{dx} = 27 - \frac{3x^2}{4}$$

$$\frac{d^2V}{dx^2} = \frac{-6x}{4} = \frac{-3x}{2}$$

$$\frac{dV}{dx} = 0 \Rightarrow 27 - \frac{3x^2}{4} = 0$$

$$\Rightarrow \quad 3x^2 = 108$$

$$\Rightarrow \quad x^2 = 36 \Rightarrow x = 6$$

at $x = 6$, $\frac{d^2V}{dx^2}$ is –ve

$\Rightarrow \quad x = 6$ is a maximum point.

at $x = 6, y = \frac{108-36}{4\times 6} = \frac{72}{24} = 3$

so x = 6 cm and y = 3 cm

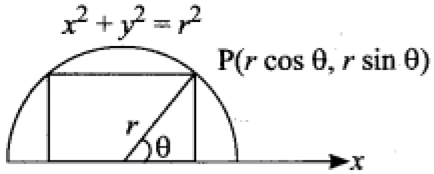
Question 9.

Find the dimensions of the largest rectangle that can be inscribed in a semi-circle of radius r cm.

Solution:

Let θ be the angle made by OP with the positive direction of the x-axis.

Then the area of rectangle A is



$A(\theta) = (2\, r \cos\theta)(r \sin\theta)$
$= r^2\, 2 \sin\theta \cos\theta = r^2 \sin 2\theta$
Now $A(\theta)$ is maximum when $\sin 2\theta$ is maximum.
The maximum value of

$\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{4}$. (Note that $A'(\theta) = 0$ when $\theta = \frac{\pi}{4}$)

Therefore the critical number is $\frac{\pi}{4}$. The Area $A\left(\frac{\pi}{4}\right) = r^2$.

Question 10.
A manufacturer wants to design an open box having a square base and a surface area of 108 sq. cm. Determine the dimensions of the box for the maximum volume.
Solution:
Let the side of the square base be = x cm and the height be = y cm
Surface area = 108 sq cm

$\Rightarrow x^2 + 4xy = 108$ sq cm .... (1)

Volume $= x^2 y$ .... (2)

from (1) $\Rightarrow 4\,xy = 108 - x^2$

$\Rightarrow \quad y = \frac{108 - x^2}{4x}$ .... (3)

Substituting (3) in (2) we get

$$V = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}$$

$$V = 27x - \frac{x^3}{4}$$

$$\frac{dV}{dx} = 27 - \frac{3x^2}{4}$$

$$\frac{d^2V}{dx^2} = \frac{-6x}{4} = \frac{-3x}{2}$$

$$\frac{dV}{dx} = 0 \Rightarrow 27 - \frac{3x^2}{4} = 0$$

$$\Rightarrow 3x^2 = 108$$

$$\Rightarrow x^2 = 36 \Rightarrow x = 6$$

$$\text{at } x = 6, \frac{d^2V}{dx^2} \text{ is } -\text{ve}$$

$$\Rightarrow x = 6 \text{ is a maximum point.}$$

$$\text{at } x = 6, y = \frac{108-36}{4\times 6} = \frac{72}{24} = 3$$

so x = 6 cm and y = 3 cm

so x = 6 cm and y = 3 cm

Question 11.
The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if r + h = 6.

Solution:
$V = \pi r^2 h$
Given $r + h = 6 \Rightarrow h = 6 - r$
$V = \pi r^2(6 - r) = 6\pi r^2 - \pi r^3$

$$\frac{dV}{dr} = 6\pi(2r) - 3\pi r^2$$

$$= 12\pi r - 3\pi r^2$$

$$\frac{d^2V}{dr^2} = 12\pi - 6\pi r$$

$$\frac{dV}{dr} = 0 \Rightarrow 12\pi r - 3\pi r^2 = 0$$

$3\pi r(4 - r) = 0 \Rightarrow r = 0$ or 4
when r = 4, h = 2
So $v = \pi(16)(2) = 32\pi$
when r = 0, V = 0
So the maximum volume = 32π and the minimum volume = 0

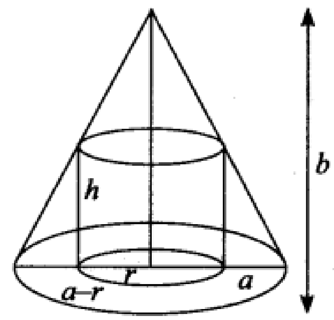
Question 12.
A hollow cone with a base radius of a cm and a height of b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times the volume of the cone.

Solution:
The height of cone = h = b
The base radius = r = a
The base radius of cylinder = r

The height of cylinder = h

$$\text{From the diagram} \Rightarrow \frac{h}{a-r} = \frac{b}{a} \quad \text{(using similar triangles)}$$

$$\Rightarrow h = \frac{b}{a}(a-r) = b - \frac{b}{a}r$$



$$\text{Volume of cylinder } V = \pi r^2 h$$

$$= \pi r^2 \left[b - \frac{b}{a}r\right]$$

$$(i.e.,) \quad V = \pi b r^2 - \frac{\pi b}{a}r^3$$

$$\frac{dV}{dr} = 2\pi b r - \frac{\pi b}{a}(3r^2)$$

$$= \frac{2\pi abr - 3\pi br^2}{a}$$

$$\frac{dV}{dr} = 0 \Rightarrow \pi br\,(2a - 3r) = 0 \Rightarrow 2a = 3r \Rightarrow r = \frac{2a}{3}$$

$$\frac{d^2V}{dr^2} = 2\pi b - \frac{6\pi br}{a}$$

$$\text{at } r = \frac{2a}{3},\ \frac{d^2V}{dr^2} = 2\pi b - \frac{6\pi b}{a}\left(\frac{2a}{3}\right) = -2\pi b$$

$r = \frac{2a}{3}$ is a maximum point

So volume is maximum at $r = \frac{2a}{3}$

$$\text{So} \quad h = b - \frac{b}{a}\left(\frac{2a}{3}\right) = \frac{b}{3}$$

$$\text{Volume of cylinder} = \pi r^2 h = \pi\left(\frac{2a}{3}\right)^2 \left(\frac{b}{3}\right)$$

$$= \pi\left(\frac{4a}{a}\right)^2 \left(\frac{b}{3}\right) = \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$$

$$= \frac{4}{9}\,\text{(volume of cone)}$$

Ex 7.9

Question 1.

Find the asymptotes of the following curves:

(i) $f(x) = \dfrac{x^2}{x^2-1}$ (ii) $f(x) = \dfrac{x^2}{x+1}$ (iii) $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

(iv) $f(x) = \dfrac{x^2-6x-1}{x+3}$ (v) $f(x) = \dfrac{x^2+6x-4}{3x-6}$

Solution:

(i) $\lim\limits_{x \to 1^-} \dfrac{x^2}{x^2-1} = -\infty$ and $\lim\limits_{x \to 1^+} \dfrac{x^2}{x^2-1} = \infty$

So $x = -1$ and $x = 1$ are vertical asymptotes.

$$\text{as } \lim_{x \to \infty} \frac{x^2}{x^2-1} = \lim_{x \to \infty} \frac{x^2/x^2}{1-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1}{1-\frac{1}{x^2}} = 1$$

y = 1 is a horizontal asymptote

So the asymptotes are x = -1, x = +1, y = 1

(ii) Since the numerator is of higher degree than the denominator we have a slant asymptote to find that asymptote we have to divide the numerator by the denominator So the slant asymptote is y = x – 1

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & -x^2+x \\ & (-)\quad(-) \\ \hline & -x \\ & -x-1 \\ & (+)\quad(+) \\ \hline & +1 \end{array}$$

Also $\lim\limits_{x \to -1} \dfrac{x^2}{x+1} = \infty$

So $x = -1$ is a vertical asymptote

$$(iii) \quad \lim_{x \to \infty^+} \frac{3x}{\sqrt{x^2+2}} = 3 \lim_{x \to \infty} \frac{x}{\sqrt{x^2+2}}$$

$$= 3 \lim_{x \to \infty} \frac{x}{x \cdot \frac{1}{x}\sqrt{x^2+2}} = 3 \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{2}{x^2}}}$$

$$= \frac{3}{\lim_{x \to \infty}\sqrt{1+\frac{2}{x^2}}} = \frac{3}{\sqrt{\lim_{x \to \infty}\left(1+\frac{2}{x^2}\right)}}$$

$$= \frac{3}{1} = 3$$

$$\text{RHL}: \lim_{x \to \infty^+} \frac{3x}{\sqrt{x^2+2}} = 3 \lim_{x \to \infty} \frac{x}{\sqrt{x^2+2}} = 3$$

$$\text{LHL}: \lim_{x \to \infty^-} \left(\frac{-3x}{\sqrt{x^2+2}}\right) = -3 \lim_{x \to \infty} \frac{x}{\sqrt{x^2+2}} = -3$$

∴ y = 3 and y = -3 are the horizontal asymptotes and there is no slant asymptote

(iv) Since the numerator is of highest degree than the denominator. We have a slant asymptote to find it we have to divide numerator by the denominator.

$$\begin{array}{r|l} x+3 & x^2 - 6x + 1 \;\; (x-9 \\ & x^2 + 3x \\ & (-) \;\; (-) \\ \hline & -9x + 1 \\ & -9x - 27 \\ & (+) \;\; (+) \\ \hline & 28 \end{array}$$

So the equation of asymptotes is y = x – 9 and x = -3

(v) Since the numerator is of the highest degree than the denominator.

We have a slant asymptote to find it we have to divide the numerator by the denominator.

$$\begin{array}{r|l} 3x-6 & x^2 + 6x - 4 \;\; \left(\frac{x}{3} + \frac{8}{3}\right. \\ & x^2 - 2x \\ & (-) \;\; (+) \\ \hline & +\, 8x - 4 \\ & (+)\, 9x - 16 \\ & (-) \;\; (+) \\ \hline & 12 \\ \hline \end{array}$$

So the equation of asymptote is $y = \frac{x}{3} + \frac{8}{3}$
and $3x - 6 = 0$
$\Rightarrow x = 2$

Question 2.
Sketch the graphs of the following functions:

(i) $y=-\frac{1}{3}\left(x^3-3x+2\right)$ (ii) $y=x\sqrt{4-x}$ (iii) $y=\frac{x^2+1}{x^2-4}$

(iv) $y=\frac{1}{1+e^{-x}}$ (v) $y=\frac{x^3}{24}-\log x$

Solution:

(i) $y=-\frac{1}{3}\left(x^3-3x+2\right)$

Factorizing we get

$y=-\frac{1}{3}(x-1)^2(x+2)=f(x)$

• The domain and the range of the given function f(x) are the entire real line.
• Putting y = 0 we get x = 1, 1, -2.
Hence the x intercepts are (1, 0) and (-2, 0) and by putting x = 0 we get $y=-\frac{2}{3}$

$\therefore$ The y intercept is $\left(0,-\frac{2}{3}\right)$

• $f'(x)=-\frac{(3x^2-3)}{3}=-(x^2-1)=(1-x^2)$

$f'(x)=0 \Rightarrow 1-x^2=0 \Rightarrow x=\pm 1$

the critical points of the curve occur at $x=\pm 1$

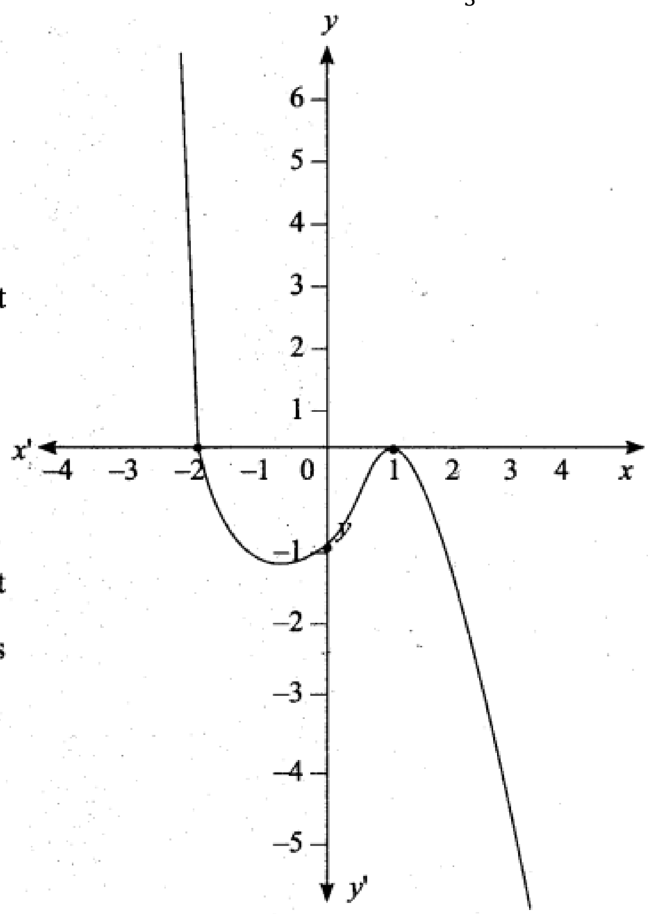
• $f''(x)=-2x$

$f''(1)=-2<0 \therefore f(x)$ is maximum at $x=1$

and local maximum is $f(1)=0$

$f''(-1)=2>0 \Rightarrow f(x)$ is minimum at $x=-1$ and the local minimum is $f(-1)=-\frac{4}{3}$

• $f''(x)=-2x<0 \;\forall\; x>c$



$\therefore$ The function is concave upward in the negative real line.
• Since f''(x) = 0 at x = 0 and f''(x) changes its sign when passing through x = 0, x = 0 is a

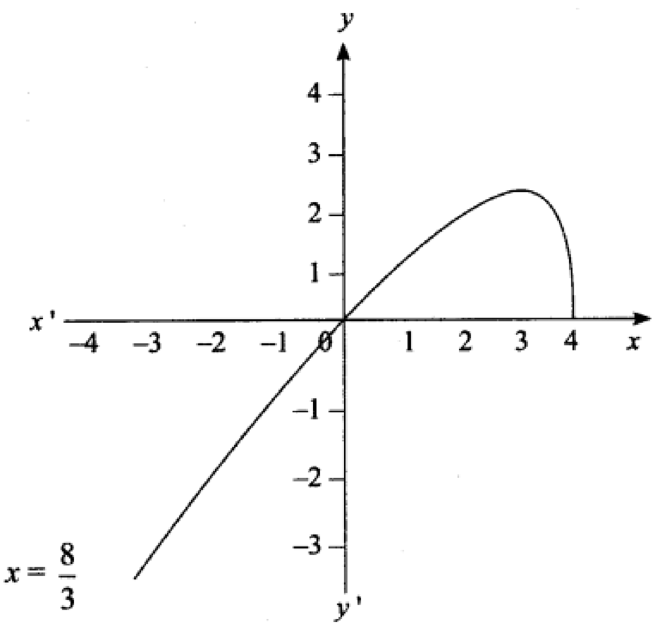
point of inflection is $(0, -\frac{2}{3})$

• The curve has no asymptotes.

(ii)

$$y = x\sqrt{4-x} = f(x)$$

(*ii*) $y = x\sqrt{4-x} = f(x)$



- When $x > 4$ the curve doesnot exists and it exists for $x \le 4$

  The domain is $[-\infty, 4]$ and the range is $\left(-\infty, \frac{16}{3\sqrt{3}}\right]$.
- The curve passes through the origin. The curve intersect x axis at (4, 0).
- $f'(x) = \frac{-x}{2\sqrt{4-x}} + \sqrt{4-x} = \frac{8-3x}{2\sqrt{4-x}}$

  $\therefore\ f'(x) = 0 \Rightarrow 8 - 3x = 0 \Rightarrow x = \frac{8}{3}$

  $\therefore$ Critical point of the curve occur at $x = \frac{8}{3}$
- $f''(x) = \frac{3x-16}{4(4-x)^{3/2}}$

  $f''\left(\frac{8}{3}\right) = -\frac{3\sqrt{3}}{4} < 0 \Rightarrow f(x)$ is maximum at $x = \frac{8}{3}$

  and the local maximum is $f\left(\frac{8}{3}\right) = \frac{16}{3\sqrt{3}}$ and the local minimum is 0 at $x = 4$ (from the graph)
- $f''(x) = \frac{3x-16}{4(4-x)^{3/2}} < 0\ \forall\ x < 4$

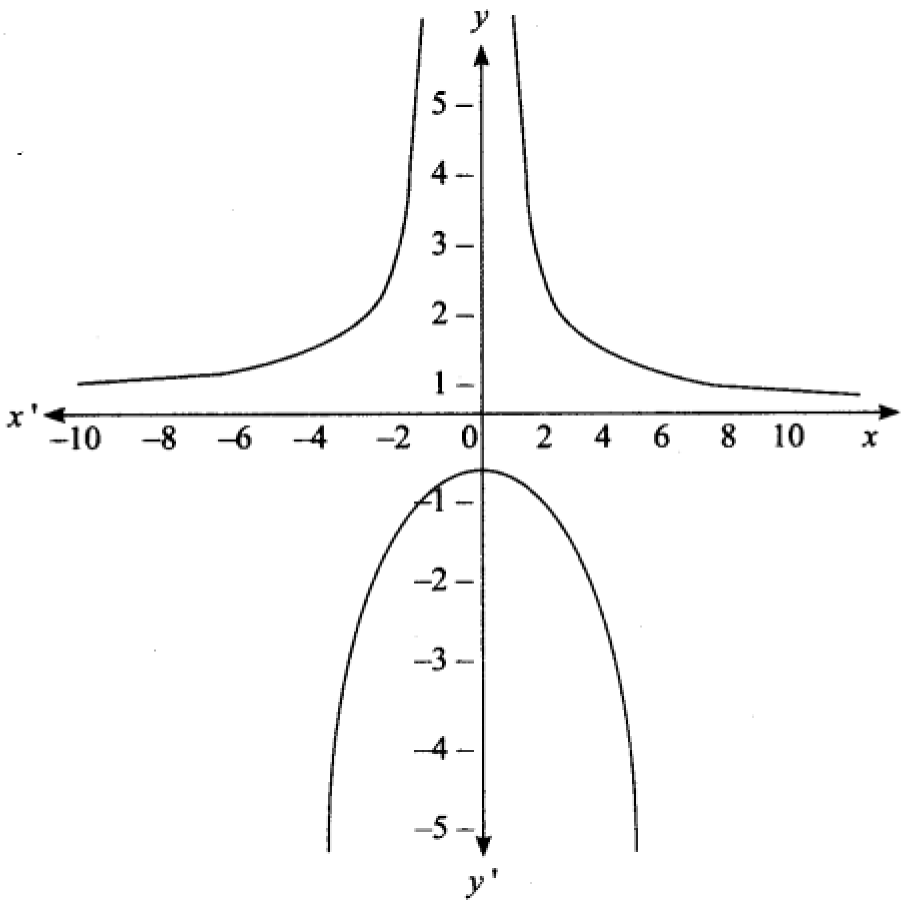
The curve concave downward in the negative real line

• No point of inflection exists.

• as $x \to \infty$, $y = \pm\infty$ and so the curve does not have any asymptotes

(iii)

$$y = \frac{x^2+1}{x^2-4}$$



- The domain of the given function $f(x)$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
  (*i.e.*,) $x < -2$ or $-2 < x < 2$ or $x > 2$
  Range of $f(x)$ is
  $\left(-\infty, \frac{-1}{4}\right] \cup (1, \infty)$
  (*i.e.*,) $f(x) \leq \frac{-1}{4}$ or $f(x) > 1$
- Putting y = 0. x is unreal hence there is no 'x' intercept. By putting x = 0 we get

$$y = \frac{-1}{4}$$

y intercept is $\left(0, \frac{-1}{4}\right)$.

- $f'(x) = \dfrac{-10x}{\left(x^2 - 4\right)^2}$

  $\therefore\ f'(x) = 0 \Rightarrow x = 0.$

  $\therefore$ The critical point is at $x = 0$

- $f''(x) = \dfrac{10\left(x^2 - 4\right)\left(3x^2 + 4\right)}{\left(x^2 - 4\right)^4}$

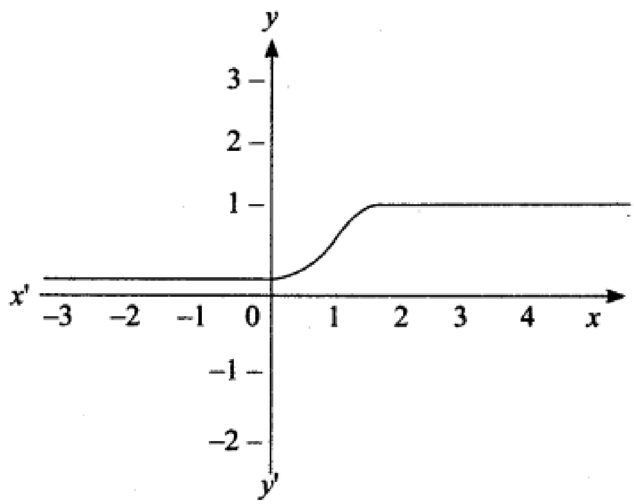
  $f''(0) = \dfrac{-5}{8} < 0.\ \therefore\ f(x)$ is maximum at $x = 0$

  and the local maximum is $f(0) = -\dfrac{1}{4}$

• No points of reflection

• When x = ± 2, y = ∞, Vertical asymptotes are x = 2 and x = -2 and horizontal asymptote is y = 1

(iv) $y = \dfrac{1}{1 + e^{-x}} = f(x)$



- The domain of the function $f(x)$ is the entire real line

  (*i.e.,*) $(-\infty, \infty)$ (*i.e.,*) $-\infty < x < \infty$ and the range is $(0, 1)$ (*i.e.,*) $0 < f(x) < 1$

- No 'x' intercept for $f(x)$ and when $x = 0, y = \dfrac{1}{2}$

  $\therefore$ The y intercept is $(0, \dfrac{1}{2})$

- $f'(x) = \dfrac{e^{-x}}{\left(1 + e^{-x}\right)^2}$

f'(x) = 0 ⇒ e^{-x} = 0 which is not possible hence there is no extremum.

• No vertical asymptote for the curve and the horizontal asymptotes are y = 1 and y = 0

(v)

$$y = \frac{x^3}{24} - \log x = f(x)$$

• The curve exists only for positive values of (x > 0)

$(i.e.,)$ domain is $(0, \infty)$ and the range is $\left(\frac{1}{3} - \log e^2 > \infty\right)$

• No intersection points are possible

• $f'(x) = \frac{x^2}{8} - \frac{1}{x}$

$(i.e.,)$ domain is $(0, \infty)$ and the range is $\left(\frac{1}{3} - \log e^2 > \infty\right)$

• No intersection points are possible

• $f'(x) = \frac{x^2}{8} - \frac{1}{x}$

$f'(x) = 0 \Rightarrow x^3 - 8 = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$

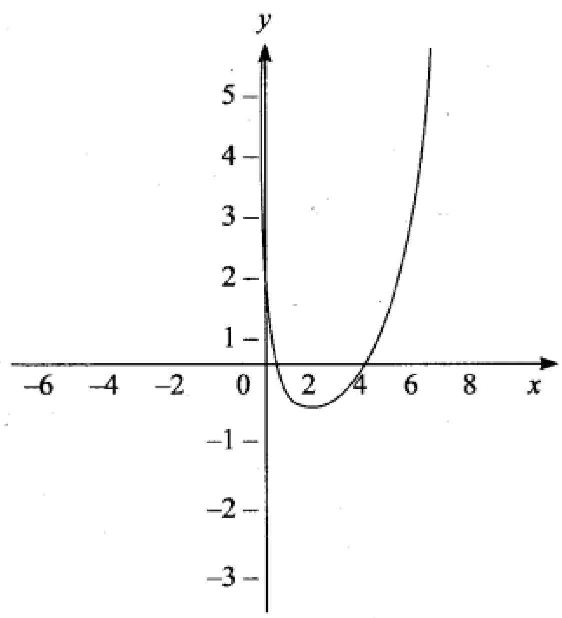
$\therefore$ Critical point occur at $x = 2$

• $f''(x) = \frac{x}{4} + \frac{1}{x^2}$

$f''(2) = \frac{3}{4} > 0$

$\therefore$ $f(x)$ is minimum at $x = 2$ and

the local minimum is $f(2) = \frac{1}{3} - \log e^2$



• No point of inflection.

• No horizontal asymptote is possible.

But the vertical asymptote is x = 0(y-axis).

Ex 7.10

Question 1.
The volume of a sphere is increasing in volume at the rate of 3πcm3/sec. The rate of change of its radius when radius is $\frac{1}{2}$ cm ....
(a) 3 cm/s
(b) 2 cm/s
(c) 1 cm/s
(d) $\frac{1}{2}$ cm/s
Solution:
(a) 3 cm/s
Hint:

Hint: $v = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

Given $\frac{dv}{dt} = 3\pi$ and $r = \frac{1}{2}$ we get

$$3\pi = \frac{4}{3}\pi(3)\left(\frac{1}{4}\right)\frac{dr}{dt}$$

$$\frac{dr}{dt} = 3$$

Question 2.
A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.

(1) $\frac{3}{25}$ radians/sec (2) $\frac{4}{25}$ **radians/sec** (3) $\frac{1}{5}$ radians/sec (4) $\frac{1}{3}$ radians/sec

Solution:
(b) $\frac{4}{25}$ radian/sec
Hint:

(1) $\frac{3}{25}$ radians/sec (2) $\frac{4}{25}$ **radians/sec** (3) $\frac{1}{5}$ radians/sec (4) $\frac{1}{3}$ radians/sec

$$(i.e.,)\ 10 = 40\left(\frac{50}{40}\right)^2 \frac{d\theta}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{10\times 40}{2500} = \frac{4}{25} \text{ radians / sec}$$

Question 3.
The position of a particle moving along a horizontal line of any time t is given by s(t) = 80t – 16t2. The time at which the particle is at rest is …….

(1) $t=0$ **(2)** $t=\frac{1}{3}$ (3) $t=1$ (4) $t=3$

Solution:
(2) $t = \frac{1}{3}$
Hint:

$$s(t) = 3t^2 - 2t - 8$$

$$v = \frac{ds}{dt} = 6t - 2$$

$$v = 0 \Rightarrow 6t - 2 = 0 \Rightarrow t = 2/6 = \frac{1}{3}$$

Question 4.
A stone is thrown up vertically. The height reaches at time t seconds is given by x = 80t – 16t2. The stone reaches the maximum height in time t seconds is given by …..
(a) 2
(b) 2.5
(c) 3
(d) 3.5
Solution:
(b) 2.5
Hint:

$$x = 80t - 16t^2$$

$$\frac{dx}{dt} = 80 - 32t$$

$$\frac{dx}{dt} = 0 \Rightarrow 80 = 32t$$

$$t = \frac{80}{32} = \frac{5}{2} = 2.5$$

Question 5.

Find the point on the curve $6y = x^3 + 2$ at which y-coordinate changes 8 times as fast as x-coordinate is …….

(a) (4, 11)
(b) (4, -11)
(c) (-4, 11)
(d) (-4, -11)

Solution:

(a) (4, 11)

Hint:

$$6y = x^3 + 2$$
$$6\frac{dy}{dx} = 3x^2$$
$$(i.e.,)\ 6(8) = 3x^2$$
$$\Rightarrow x^2 = \frac{48}{3} = 16$$
$$\Rightarrow x = \pm 4$$

when $x = 4$

$$6y = 66$$
$$\Rightarrow y = 11$$

when $x = -4$

$$\Rightarrow 6y = -62$$
$$\Rightarrow y = \frac{-62}{6} = \frac{-31}{3}$$

$\therefore$ The points are (4, 11) and $\left(-4, -\frac{31}{3}\right)$

Question 6.

The abscissa of the point on the curve f(x) = $\sqrt{8-2x}$ at which the slope of the tangent is -0.25?

(a) -8
(b) -4
(c) -2
(d) 0

Solution:

(b) -4

Hint:

$$f(x) = \sqrt{8-2x}$$

$$f'(x) = \frac{1}{2\sqrt{8-2x}}(-2) = -\frac{1}{4}$$

$$\Rightarrow \quad \frac{-1}{\sqrt{8-2x}} = \frac{-1}{4}$$

$$\Rightarrow \quad \sqrt{8-2x} = 4 \Rightarrow 8-2x = 16$$

$$2x = 8-16 = -8$$

$$x = -4$$

Question 7.
The slope of the line normal to the curve f(x) = 2cos 4x at x = π2 is ……..

(a) $-4\sqrt{3}$ (b) -4 (c) $\frac{\sqrt{3}}{12}$ (d) $4\sqrt{3}$

Solution:
(c) $\frac{\sqrt{3}}{12}$

Hint:

$$f(x) = 2\cos 4x$$

$$f'(x) = 2[-\sin 4x](4)$$

$$= -8\sin 4x$$

$$\text{at } f'(x) \text{ at } x = \frac{\pi}{12} = -8\sin\frac{\pi}{3}$$

$$= -8\frac{\sqrt{3}}{2} = -4\sqrt{3} = m$$

$$\text{So slope of the normal} = \frac{-1}{m} = \frac{-1}{-4\sqrt{3}}$$

$$= \frac{1}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{12}$$

Question 8.
The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when

(a) $y = 0$ (b) $y = \pm \sqrt{3}$ (c) $y = \frac{1}{2}$ (d) $y = \pm\sqrt{3}$

Solution:
(b) $y=\pm\sqrt{3}$
Hint:

$$y^2 - xy + 9 = 0$$

Differentiating w.r. to y

$$2y - x(1) - y\frac{dx}{dt} = 0$$

$$\therefore \frac{dx}{dt} = \frac{2y - x}{y}$$

The tangent is vertical

$$\Rightarrow x = c$$

$$\text{so } \frac{dx}{dt} = 0 \; (i.e.,) \; \frac{2y - x}{y} = 0$$

$$\Rightarrow x = 2y$$

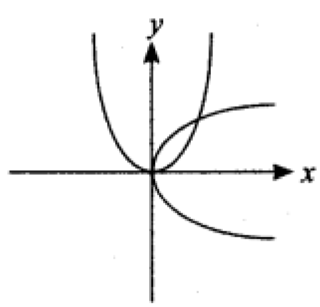
Substituting $x = 2y$ in the curve

$$y^2 - 4y^2 + 9 = 0 \Rightarrow y^2 = 3$$

$$\Rightarrow y = \pm\sqrt{3}$$

Question 9.
Angle between $y^2 = x$ and $x^2 = y$ at the origin is ……

(a) $\tan^{-1}\frac{3}{4}$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$



Solution:
(c) $\frac{\pi}{2}$
Hint:
The angle between the parabolas is the angle between the axes $= \frac{\pi}{2}$

Question 10.

What is the value of the limit $\lim_{x \to 0}\left(\cot x - \frac{1}{x}\right)$?

(a) 0
(b) 1
(c) 2
(d) ≤
Solution:
(a) 0
Hint:

$$\lim_{x \to 0} \cot x - \frac{1}{x} = \frac{\cos x}{\sin x} - \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{x\sin x} = \frac{0}{0} \text{ form}$$

Applying L.H. rule

$$\lim_{x \to 0} \frac{x(-\sin x) + \cos x(1) - \cos x}{x\cos x + \sin x} = \frac{-x\sin x}{x\cos x + \sin x} = \frac{0}{0}$$

Again applying L.H. Rule

$$\lim_{x \to 0} -\left[\frac{x\cos x + \sin x(1)}{x(-\sin x) + \cos x + \cos x}\right] = \frac{0}{2} = 0$$

Question 11.
The function $\sin^4 x + \cos^4 x$ is increasing in the interval

(a) $\left[\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (b) $\left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$ (c) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left[0, \frac{\pi}{4}\right]$

Solution:
(c) $[\frac{\pi}{4}, \frac{\pi}{2}]$
Hint:

$$
\begin{aligned}
f(x) &= \sin^4 x + \cos^4 x \\
f'(x) &= 4\sin^3 x \cos x + 4\cos^3 x(-\sin x) \\
&= 4\sin x \cos x(\sin^2 x - \cos^2 x) \\
f'(x) &= 0 \Rightarrow \sin x = 0 \text{ (or) } \cos x = 0 \text{ (or) } \sin^2 x - \cos^2 x = 0 \\
&\qquad x = 0 \text{ (or) } x = \frac{\pi}{2} \text{ (or) } x = \frac{\pi}{4}
\end{aligned}
$$

In $[0, \frac{\pi}{4}]$ $f'(x)$ is –ve so $f(x)$ is decreasing.

In $[\frac{\pi}{4}, \frac{\pi}{2}]$, $f'(x)$ is +ve so $f(x)$ is increasing.

Question 12.
The number given by the Rolle's theorem for the function x3 – 3x2, x ∈ [0, 3] is ………
(a) 1
(b) √2
(c) $\frac{3}{2}$
(d) 2
Solution:
(d) 2
Hint:

(a) 1 (b) $\sqrt{2}$ (c) $\frac{3}{2}$ **(d) 2**

Question 13.
The number given by the Mean value theorem for the function $\frac{1}{x}$, x ∈ [1, 9] is …..
(a) 2
(b) 2.5
(c) 3
(d) 3.5
Solution:
(c) 3
Hint:

$$f(x) = \frac{1}{x}$$

$$f'(x) = \frac{-1}{x^2} \quad \text{....(1)}$$

Here $a = 1,\ b = 9$

So $$\frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{9} - 1}{9 - 1} = \frac{\frac{-8}{9}}{8} = \frac{-1}{9} \quad \text{....(2)}$$

from (1) and (2)

$$-\frac{1}{x^2} = -\frac{1}{9} \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

but $x = 3 \in [1, 9]$

Question 14.
The minimum value of the function |3 – x | + 9 is ....
(a) 0
(b) 3
(c) 6
(d) 9
Solution:
(d) 9
Hint:
f(x) = |3 – x| + 9
Minimum value of |3 – x| = 0
Minimum value of |3 – x| + 9 = 0 + 9 = 9

Question 15.
The maximum slope of the tangent to the curve y = e^x sin x, x ∈ [0, 2n] is at ....
(a) $x = \frac{\pi}{4}$ (b) $x = \frac{\pi}{2}$ (c) $x = \pi$ (d) $x = \frac{3\pi}{2}$
Solution:
(b) $x=\frac{\pi}{2}$
Hint:

$$
\begin{aligned}
y &= e^x \sin x \\
\frac{dy}{dx} &= e^x (\cos x) + \sin x\,(e^x) \\
&= e^x (\sin x + \cos x) = m \text{ (say)} \\
\text{Now } \frac{dm}{dx} &= e^x [\cos x - \sin x] + [\sin x + \cos x]\, e^x
\end{aligned}
$$

$$
\begin{aligned}
&= e^x [\cos x - \sin x + \sin x + \cos x] \\
&= 2 \cos x\, e^x \\
\frac{dm}{dx} &= 0 \Rightarrow \cos\ x = 0 \Rightarrow x = \frac{\pi}{2}
\end{aligned}
$$

Question 16.
The maximum value of the function $x^2 e^{-2x}$, $x > 0$ is ……..

(a) $\frac{1}{e}$ (b) $\frac{1}{2e}$ (c) $\frac{1}{e^2}$ (d) $\frac{4}{e^4}$

Solution:
(c) $\frac{1}{e2}$
Hint:

$$
\begin{aligned}
y &= x^2 e^{-2x} \\
\frac{dy}{dx} &= x^2 (e^{-2x})(-2) + e^{-2x}(2x) \\
&= e^{-2x} [-2x^2 + 2x] \\
y' &= 0 \Rightarrow -2x^2 + 2x = 0 \\
&\Rightarrow 2x(1-x) = 0 \Rightarrow x = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{at } x = 0, \quad y &= 0 \\
\text{at } x = 1, \quad y &= \frac{1}{e^2}
\end{aligned}
$$

Question 17.
One of the closest points on the curve $x^2 - y^2 = 4$ to the point (6, 0) is ……..

(a) (2, 0) (b) $(\sqrt{5}, 1)$ (c) $(3, \sqrt{5})$ (d) $(\sqrt{13}, -\sqrt{3})$

Solution:
(c) (3,5–√)
Hint:

Given $x^2 - y^2 = 4$, point (6, 0)

Any point on the curve is $\left(x, \pm\sqrt{x^2-4}\right)$

Distance between the points is $\sqrt{(x-6)^2 + x^2 - 4}$

Substituting all the given options we get minimum distance point (3,5–√)

Question 18.
The maximum product of two positive numbers, when their sum of the squares is 200, is ……..
(a) 100
(b) 25√7
(c) 28
(d) 24√14
Solution:
(a) 100

Hint: Let the positive numbers be x, y

Here $x^2 + y^2 = 200 \Rightarrow y^2 = 200 - x^2 \Rightarrow y = \sqrt{200 - x^2}$

$$\text{Now } f = xy = x\sqrt{200 - x^2}$$

$$= \sqrt{200x^2 - x^4}$$

$$\frac{df}{dx} = \frac{1}{2\sqrt{200x^2 - x^4}}(400x - 4x^3)$$

$$\frac{df}{dx} = 0 \Rightarrow 400x - 4x^3 = 0$$

$$\Rightarrow 4x(100 - x^2) = 0$$

$$\Rightarrow x = 0 \text{ or } \pm 10$$

Here $x = 10 \Rightarrow y = \sqrt{200 - 100} = 10$

Product is $xy = (10)(10) = 100$

Question 19.
The curves = $ax^4 + bx^2$ with $ab > 0$
(a) has no horizontal tangent
(b) is concave up
(c) is concave down
(d) has no points of inflection
Solution:
(d) has no points of inflection

Question 20.
The point of inflection of the curve y = $(x - 1)^3$ is ……….
(a) (0, 0)
(b) (0, 1)
(c) (1, 0)
(d) (1, 1)
Solution:
(c) (1, 0)
Hint:

$$y = (x-1)^3$$
$$y' = 3(x-1)^2$$
$$y'' = 6(x-1)$$

$$y'' = 0 \Rightarrow 6(x-1) = 0$$
$$x = 1$$
$$\text{at } x = 1,\ y = 0$$
$$\therefore \text{ point is } (1, 0)$$