The goal of this calculation is to prove the complex integral transform of the Stirling operator ϕ:

$$\phi f(x) = \frac{\Gamma(x+1)}{2i\pi} \oint_\gamma \frac{e^z\, f(z)}{z^{x+1}} dz, \tag{1}$$

We begin with Cauchy's integral formula, which we can write

$$f(x) = \frac{1}{2i\pi} \oint_\gamma \frac{f(z)}{z-x} dz, \tag{2}$$

Now notice that if we replace $f(z)$ with $e^{z-x}f(z)$ the formula

$$f(x) = \frac{1}{2i\pi} \oint_\gamma \frac{e^{z-x}\, f(z)}{z-x} dz, \tag{3}$$

is still valid because we then have

$$f(x) = e^{-x} \frac{1}{2i\pi} \oint_\gamma \frac{e^z\, f(z)}{z-x} dz,$$

and using Eqn. 2 this reduces to

$$\begin{aligned} f(x) &= e^{-x}\, e^x f(x) \\ f(x) &= f(x) \end{aligned}$$

Now let us start again with Eqn. 3

$$f(x) = \frac{1}{2i\pi} \oint_\gamma \frac{e^{z-x}\, f(z)}{z-x} dz,$$

Applying ϕ to both sides results in

$$\phi f(x) = \phi \frac{1}{2i\pi} \oint_\gamma \frac{e^{z-x}\, f(z)}{z-x} dz,$$

and because of linearity of ϕ

$$\phi f(x) = \frac{1}{2i\pi} \oint_\gamma e^z\, f(z)\, \phi\left(\frac{e^{-x}}{z-x}\right) dz,$$

Comparing this to Eqn. 1, we now only have to show that

$$\phi\left(\frac{e^{-x}}{z-x}\right) = \frac{\Gamma(x+1)}{z^{x+1}}$$

While this is pretty difficult to show, the inverse

$$\frac{e^{-x}}{z-x} = \phi^{-1}\left(\frac{\Gamma(x+1)}{z^{x+1}}\right)$$

is much easier to prove, using the definition of ϕ^{-1}

$$\phi^{-1} f(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} f(k)$$

Substituting $f(x) = \frac{\Gamma(x+1)}{z^{x+1}}$ we arrive at

$$\begin{aligned}
\phi^{-1}\left(\frac{\Gamma(x+1)}{z^{x+1}}\right) &= e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{\Gamma(k+1)}{z^{k+1}} \\
&= \frac{e^{-x}}{z} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{k!}{z^k} \\
&= \frac{e^{-x}}{z} \sum_{k=0}^{\infty} \left(\frac{x}{z}\right)^k \\
&= \frac{e^{-x}}{z} \frac{1}{1 - \frac{x}{z}}
\end{aligned}$$

And finally

$$\phi^{-1}\left(\frac{\Gamma(x+1)}{z^{x+1}}\right) = \frac{e^{-x}}{z-x} \qquad \text{q.e.d.}$$