Big Phi

This is a collection of basic results about Φ. These notes assume the reader is familiar with the operators $\mathcal{D}$, Δ, $\mathcal{T}$ and ϕ; see the ϕ notes for information on these.

List of identities:

$$\Phi(A) = \phi A \phi^{-1}$$

$$\Phi^n(\Phi^m(A)) = \Phi^{n+m}(A)$$

$$\Phi^n(\phi^k) = \phi^k$$

$$\Phi(\mathcal{D}^k) = \Delta^k$$

$$\Phi^n(A^k) = (\Phi^n(A))^k$$

$$\Phi^n(A+B) = \Phi^n(A) + \Phi^n(B)$$

$$\Phi^n(aA) = a\Phi^n(A)$$

$$\Phi^n(p(\mathcal{D})) = p(\Phi^n(\mathcal{D}))$$

$$\exp(\Phi^n(A)) = \Phi^n(\exp(A))$$

$$\ln(\Phi^n(A)) = \Phi^n \ln(A)$$

$$\exp(\Phi^n(\mathcal{D})) = \Phi^n(\mathcal{T})$$

$$\ln(\Phi^{n+1}(\mathcal{D}) + 1) = \Phi^n(\mathcal{D}) = \exp(\Phi^{n-1}(\mathcal{D})) - 1$$

$$\ln(\Phi^{n+1}(\mathcal{T})) + 1 = \Phi^n(\mathcal{T}) = \exp(\Phi^{n-1}(\mathcal{T}) - 1)$$

$$\left.\begin{array}{ll} \Phi(D) & = e^D - 1 \\ \Phi^{-1}(D) & = \ln(D+1) \\ \Phi(T) & = e^{T-1} \\ \Phi^{-1}(T) & = \ln(eT) \end{array}\right\} D = \Phi^n(\mathcal{D}),\ T = \Phi^n(\mathcal{T})$$

Define the function $\Phi(A) = \phi A \phi^{-1}$ on an arbitrary linear operator A over $\mathbb{C}[x]$. It's helpful to remember that $\Phi^n(\Phi^m(A)) = \Phi^{n+m}(A)$, i.e. composing powers of Φ is isomorphic to integer additon.

We begin by introducing the motivating identity:

Delta Theorem

$$\Phi(\mathcal{D}) = \Delta$$

Proof. This is simply due to ϕ's conjugation property, $\Delta\phi = \phi\mathcal{D}$.

$\square$

This is generalised by the

Power Rule

$$\Phi^n(A^k) = (\Phi^n(A))^k$$

Proof. For positive integer k we have

$$\begin{aligned}(\Phi^n(A))^k &= \underbrace{(\phi^n A\phi^{-n})(\phi^n A\phi^{-n})\cdots(\phi^n A\phi^{-n})}_{k \text{ terms}} \\ &= \phi^n A^k \phi^{-n} \\ &= \Phi^n(A^k)\end{aligned}$$

and so the identity holds. On the other hand, inverting both sides gives

$$\begin{aligned}(\Phi^n(A))^{-k} &= (\phi^n A^k \phi^{-n})^{-1} \\ &= \phi^n A^{-k} \phi^{-n} \\ &= \Phi^n(A^{-k})\end{aligned}$$

and so the identity also holds for all negative integer k.

$\square$

This for example extends the delta theorem to nth derivatives and integrals, e.g. Φ maps the integral $\mathcal{D}^{-1}$ to the sum Δ^{-1}.

We'll also need a couple more basic identities:

Linearity of Φ

$$\begin{aligned}\Phi^n(A+B) &= \Phi^n(A) + \Phi^n(B) \\ \Phi^n(aA) &= a\Phi^n(A)\end{aligned}$$

Proof. This follows immediately from linearity of ϕ:

$$\begin{aligned}\Phi^n(A+B) &= \phi^n(A+B)\phi^{-n} \\ &= \phi^n A\phi^{-n} + \phi^n B\phi^{-n} \\ &= \Phi^n(A) + \Phi^n(B)\end{aligned}$$

and $\Phi^n(aA) = \phi^n aA\phi^{-n} = a\phi^n A\phi^{-n}$ for arbitrary constant a.

□

Together with the power rule, this means that for a differential operator $p(\mathcal{D})$ we have
$$\Phi^n(p(\mathcal{D})) = p(\Phi^n(\mathcal{D})).$$

Log Rule

$$\exp(\Phi^n(A)) = \Phi^n \exp(A)$$
$$\ln(\Phi^n(A)) = \Phi^n \ln(A)$$

Proof. By the power rule,

$$\begin{aligned}
\exp(\Phi^n(A)) &= \sum_{k=0}^{\infty} \frac{(\Phi^n(A))^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\Phi^n(A^k)}{k!} \\
&= \phi^n \sum_{k=0}^{\infty} \frac{A^k}{k!} \phi^{-n} \\
&= \phi^n \exp(A) \phi^{-n} \\
&= (\Phi^n \exp(A)).
\end{aligned}$$

This proves the first identity. Now taking the log of both sides,

$$\begin{aligned}
\Phi^n(A) &= \ln(\Phi^n \exp(A)) \\
\Leftrightarrow \quad \Phi^n \ln \exp(A) &= \ln(\Phi^n \exp(A)) \\
\Leftrightarrow \quad \Phi^n \ln(A) &= \ln(\Phi^n(A))
\end{aligned}$$

by replacing $\exp(A)$ with A.

□

In other words, Φ commutes with exponentials and logarithms.
We can extend these identities using the general exponential properties, e.g.

$$\exp(\Phi^n(A) + B) = \exp(B)\Phi^n \exp(A).$$

Shift Theorem

$$\exp(\Phi^n(\mathcal{D})) = \Phi^n(\mathcal{T})$$

Proof. From the general Taylor series we have

$$f(x+a) = \sum_{k=0}^{\infty} \frac{x^k \mathcal{D}^k}{k!} f(a).$$

We have a shift applied to $f(a)$ on the left, and the power series of an exponential on the right:

$$\begin{aligned} \mathcal{T}^x f(a) &= \exp(x\mathcal{D}) f(a) \\ \Leftrightarrow \quad \mathcal{T} &= \exp(\mathcal{D}). \end{aligned}$$

Thus applying the log rule with $\exp(\Phi^n(\mathcal{D}))$ completes the proof.

$\square$

In other words, we can exponentiate $\Phi^n(\mathcal{D})$ simply by replacing $\mathcal{D}$ with $\mathcal{T}$.

Tower Theorem

$$\begin{aligned}
\Phi^n(\mathcal{D}) &= \exp(\Phi^{n-1}(\mathcal{D})) - 1 \\
&= \ln(\Phi^{n+1}(\mathcal{D}) + 1) \\
&\text{and} \\
\Phi^n(\mathcal{T}) &= \exp(\Phi^{n-1}(\mathcal{T}) - 1) \\
&= (\ln \Phi^{n+1}(\mathcal{T})) + 1
\end{aligned}$$

Proof. By the delta theorem we have

$$\begin{aligned}
\Phi^{n+1}(\mathcal{D}) &= \Phi^n(\Delta) \\
&= \Phi^n(\mathcal{T}) - 1 \\
&= \exp(\Phi^n(\mathcal{D})) - 1
\end{aligned}$$

by linearity of Φ and the shift theorem.
The first two equations are both simple rearrangements of this.

Similarly,

$$\begin{aligned}
& \Phi^n(\mathcal{D}) = \Phi^{n-1}(\mathcal{T}) - 1 \\
\Leftrightarrow \quad & \Phi^n(\mathcal{T}) = \exp(\Phi^{n-1}(\mathcal{T}) - 1) \\
\Leftrightarrow \quad & \Phi^n(\mathcal{T}) = \ln \Phi^{n+1}(\mathcal{T}) + 1.
\end{aligned}$$

□

I like the funky $\mathcal{D}$-$\mathcal{T}$ symmetry with the parentheses, but the second pair of equations may also be written

$$\Phi^n(\mathcal{T}) = \frac{1}{e}\exp(\Phi^{n-1}\mathcal{T}) = \ln(e\Phi^{n-1}\mathcal{T}).$$

This theorem gives us new canonical definitions for Φ when the argument is of the form $D = \Phi^n(\mathcal{D})$ or $T = \Phi^n(\mathcal{T})$:

$$\begin{aligned}
\Phi(D) &= e^D - 1 \\
\Phi^{-1}(D) &= \ln(D + 1) \\
\Phi(T) &= e^{T-1} \\
\Phi^{-1}(T) &= \ln(eT).
\end{aligned}$$