3 Bounding falling factorials on $(0, r)$

For $k \in (-1, 1)$ it can be shown that $\phi(e^{kx})$ is well defined and that there exists global convergence. However, something strange happens for $k \geq 1$, specifically it seems like $\phi(e^{kx})$ seems to only converge on the positive reals, in this section I introduce an asymptotic upper bound on $\|X_n\|_\infty$ on an open interval $(0, r)$ and the effects it has on the convergence of certain exponentials.

3.1 Asymptotic upper bound on $(0, r)$

Whomever's reading this may notice that for sufficiently large n, $|X_n|$ has an absolute maximum at an $\epsilon \in (0, 1)$, this can be noticed by *depacking* $|X_n|$. In this section we will reserve the use of the variable ϵ_n to denote the local maximizer in $(0, 1)$ of $|X_n|$.

Proposition 1. *The maximizer ϵ_n is the multiplicative inverse of the root of the function:*

$$f(x) = 1 + \frac{1}{1-x} + \frac{1}{1-2x} + \cdots + \frac{1}{1-(n-1)x}$$

Proof. Notice the following:

$$(X_n)' = X_n \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-(n-1)} \right)$$

Clearly the falling factorial X_n is non-zero in the interval $(0, 1)$, so the derivative is zero at $x \in (0, 1)$ iff:

$$\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-(n-1)} = 0$$

Dividing by $\frac{1}{x}$ which is non-zero in the interval $(0, 1)$ and denoting $\lambda = \frac{1}{x}$ one has the following statement equivalent to the previous:

$$1 + \frac{1}{1-\lambda} + \frac{1}{1-2\lambda} + \cdots + \frac{1}{1-(n-1)\lambda} = 0$$

□

Proposition 2. *The maximizer ϵ_n is a strictly decreasing sequence, and therefore has a limit.*

Proof. Lets denote:

$$f_n(x) = 1 + \frac{1}{1-x} + \frac{1}{1-2x} + \cdots + \frac{1}{1-(n-1)x}$$

Clearly we have the following:

$$\begin{aligned} f_{n+1}(x) =& f_n(x) + \frac{1}{1-nx} \\ f_n'(x) >& 0 \end{aligned}$$

And so

$$f_{n+1}(1/\epsilon_n) = \frac{1}{1 - n/\epsilon_n}$$

If $1 - n/\epsilon_n > 0$, then clearly $1 - k\epsilon_n > 0$ for all $k \in \{1, \ldots, n\}$, and would therefore imply $f_n(1/\epsilon_n) > 0$, which is a contradiction. And therefore:

$$f_{n+1}(1/\epsilon_n) < 0$$

By monotony we then have that: $\epsilon_{n+1} < \epsilon_n$. □

Proposition 3. *The sequence $\|X_n\|_\infty$ has asymptotic upper bound, on $(0, r)$ given by:*

$$\|X_n\|_\infty < \epsilon_n \cdot e^{-\epsilon_n \gamma} \frac{(n-1)!}{(n-1)^{\epsilon_n}}$$

Proof. This proof assumes the reader is aware of the fact that $1 - x < e^{-x}$ for all x. Remember that for sufficiently large n, the global maximum of $|X_n|$ has maximizer ϵ_n and denote it ϵ for the sake of clarity, therefore:

$$\begin{aligned}
\|X_n\|_\infty =& |\epsilon| \cdot |\epsilon - 1| \cdots |\epsilon - (n-1)| \\
\|X_n\|_\infty =& (n-1)! \left(1 - \frac{\epsilon}{n-1}\right) \cdot \left(1 - \frac{\epsilon}{n-2}\right) \cdots (1 - \epsilon) \cdot \epsilon \\
\|X_n\|_\infty <& (n-1)! \cdot \epsilon \cdot e^{-\epsilon H_{n-1}}
\end{aligned}$$

Given that $H_n > \gamma + \log(n)$ we have:

$$\|X_n\|_\infty < \epsilon \cdot e^{-\epsilon\gamma} \frac{(n-1)!}{(n-1)^{\epsilon}}$$

□

We now have (probably) enough tools to prove that $\phi(e^{kx})$ is well defined as a function on $\mathbb{R}^+$, for $k \geq 1$.

Next time I write here again, I'll probably prove the well-definedness of $\phi(e^{kx})$ for $k \geq 1$, feeling kinda tired now though.